THE UNIVERSITY OF WESTERN ONTARIO

LONDON CANADA

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ECONOMICS 241 HANDOUT 2

2. THE REAL NUMBERS

2.1 The Real Numbers

2.2 Sequences

2.3 Continuity

2.4 Derivatives

2.5 The Exponential Function

2.6 Contraction Mappings

2.1 THE REAL NUMBERS

A rational number is a ratio m/n of integers, where $n \neq 0$. That is, the set Q of rationals is

$$Q = \{m/n \mid m \in Z, n \in Z, n \neq 0\}$$

Note that any finite decimal expansion gives a rational. For example, 2.178093 can be expressed as

$$2.178093 = \frac{2178093}{1000000}$$

Some rationals do not have finite decimal expansions:

$$\frac{1}{9} = .1111 \ldots$$

There is a sense in which the set of rational numbers has "holes" in it. For example, $\sqrt{2}$ is not a rational. To see this, we need the following Lemma:

LEMMA 2.1: Suppose m is an integer and m^2 is even. Then m is even.

PROOF: By contrapositive: we show that if m is not even (i.e. odd) then m^2 is not even (i.e. odd).

If m is odd, m can be expressed as

$$m = 2k + 1$$

(that is, m is an even number plus 1)

Thus $m^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$.

Consequently, m^2 is odd, being an even number plus 1. Q.E.D.

Now we show that $\sqrt{2}$ is not rational.

THEOREM 2.2: $\sqrt{2}$ is not rational.

PROOF: By contradiction. Suppose $\sqrt{2}$ is rational, that is

$$\sqrt{2} = m/n$$

we may assume that at least one of m and n is odd, since if they were both even, we could divide both by 2 (e.g. 24/26 = 12/13).

or $\quad 2 = m^2/n^2$

or $\quad 2n^2 = m^2$

It follows that m^2 is even, since it equals two times an ~~even~~ integer. By the lemma, m is even, i.e. m = 2k for an integer k.

Therefore

$$2n^2 = m^2 = (2k)^2 = 4k^2$$

or $\quad n^2 = 2k^2$

But then n^2, and hence n, is even. This contradicts our initial statement that at least one of m and n is odd. Q.E.D.

Even though $\sqrt{2}$ is not rational, we nonetheless consider it to be a number. In particular, there is a way of constructing the number, since a right triangle with sides of equal length will have a hypotenuse of length x satisfying, by the pythagorean theorem

$$x^2 = 1^2 + 1^2$$

or $\quad x^2 = 2.$

One manner of filling the "holes" in the rationals is as follows.

DEFINITION 2.1: a is an upper bound for a nonempty set $A \subseteq Q$ if $\forall q \in A$, $q \leq a$. b is a lower bound for A if $\forall q \in A$, $q \geq b$. If A has both upper and lower bounds, A is bounded.

DEFINITION 2.2: a is a least upper bound or <u>supremum</u> for a set $A \neq \phi$ if a is an upper bound, and if b is an upper bound to A, $a \leq b$. Similarly, a is a greatest lower bound or <u>infimum</u> for A if a is a lower bound for A and if b is a lower bound for A, $b \leq a$.

We shall denote a supremum for a set A as sup A, similarly an infimum for A is inf A. Note that the set of rationals

$$\{q \in Q \mid q^2 \leq 2\}$$

possesses both upper and lower bounds (2 and -2, respectively, will work), but has neither supremum or infimum, at least in the rationals, since $\sqrt{2}$ is not rational.

DEFINITION 2.3: r is a real number if r is a supremum of a bounded nonempty set of rationals. The set of real numbers is denoted R.

Thus, we see immediately that $\sqrt{2}$ is a real, since, trivially

$$\sqrt{2} = \sup \{x \mid x^2 \leq 2\}$$

In addition, since

$$\inf A = -\sup \{-x \mid x \in A\}$$

we see that infimums of sets of rationals are reals. Furthermore

THEOREM 2.3: $Q \subseteq R$

PROOF: We must show that $q \in Q \Rightarrow q \in R$. But

$$q = \sup \{q\} \in R.$$

Q.E.D.

THEOREM 2.4: Suppose $x < y$, $x,y \in R$. then $\exists q \in Q$, $x < q < y$.

PROOF: Since $x < y$, $\frac{y-x}{2} > 0$.

Therefore $\exists n \in \mathbb{N}$, $\frac{1}{n} < \frac{y-x}{2}$, by (1.70).

Let m be the largest integer satisfying

$$\frac{m}{n} \leq x$$

Then

$$x < \frac{m+1}{n} = \frac{m}{n} + \frac{1}{n} \leq x + \frac{y-x}{2} = \frac{y+x}{2} < \frac{y+y}{2} = y$$

and hence $\frac{m+1}{n}$ is our rational. Q.E.D.

Theorem 2.4 shows the rationals are "dense" in the reals, that is, there are no intervals of reals devoid of rationals. Thus, we may approximate reals as closely as we wish, but not perfectly, by rationals.

THEOREM 2.5: Let $\phi \neq A \subseteq R$ be bounded (i.e. $\exists b\ \forall x \in A\ |x| \leq b$). then sup $A \in R$.

PROOF: if $A = \{r\}$, then sup $A = r \in R$. Thus, we may presume A has at least two elements.

Let $B = \{q \in Q \mid \exists a \in A \quad a \geq q \geq -b\}$.

B is nonempty since A contains at least two reals, and a rational in between by Theorem 2.4. B is bounded by b, and hence sup B is a real by definition.

Let β = sup B.

CLAIM: β is an upper bound for A.

PROOF OF CLAIM: by contradiction, suppose β is not an upper bound.

Then $\exists a \in A$, $a > \beta$. But by Theorem 2.4, there is a $q \in Q$, $a > q > \beta$, and thus $q \in B$.

Hence $\beta \neq$ sup B, since $q > \beta$ and $q \in B$, a contradiction.

CLAIM: β is the least upper bound for A.

PROOF OF CLAIM: by contradiction, suppose α is an upper bound for A, and $\alpha < \beta$. Then $\forall q \in B$, $\exists a \in A \quad q \leq a \leq \alpha$ (by definition of B and upper bound).

Thus α is an upper bound for B, contradicting $\beta = \sup B$, the least upper bound.

These claims, put together, imply $\sup A = \sup B \in R$. Q.E.D.

Theorem 2.5 provides a fundamental property of the real numbers: any nonempty bounded set has a supremum. This crucial property will be used again and again in the following chapters. Effectively, we have shown that the reals do not have holes, like the rationals do, and for this reason the reals are referred to as the continuum.

An immediate consequence of Theorem 2.5 is that an increasing list of numbers which are all less than some bound <u>converges</u>. Let $x_1 \leq x_2 \leq x_3 \leq \ldots$ be a list of numbers, called an increasing sequence, and suppose $(\exists b)(\forall n)$ $x_n \leq b$. Then there is a number x_0 so that, for any small number $\varepsilon > 0$, there is an N and all the values of the sequence numbered greater than N are within ε of x_0. That is, eventually all of the sequence, except the first "few" terms (possibly trillions, but a small portion of the infinitely many terms), are very close to x_0.

Put in precise mathematical language;

$(\forall\ \varepsilon > 0)(\exists\ N)\ n \geq N \Rightarrow |x_n - x_0| < \varepsilon$

ε serves the role of "as close as you want" as long as that proximity is positive.

The limit of the increasing bounded sequence considered before, what it reaches as n gets very large (or, rather, what the sequence gets arbitrarily close to), in this case is easy to calculate:

$x_0 = \sup \{x_n \mid n = 1,2,3,\ldots\}$

The proof of this fact will be left as an exercise, with a hint: it is straightforwardly proved by contradiction.

Example 2.1 (Zeno's Paradox)

Achilles and a tortoise are going to run a footrace. Achilles can run twice as fast as the tortoise (must be a quick tortoise--or maybe Achilles' tendon hurts and he can't run so fast), and the tortoise will get a 1-meter head start. We suppose Achilles runs 1 meter per minute.

Fleet-footed Achilles bounds over a meter, and the tortoise has moved ahead 1/2 meter. Achilles leaps this 1/2 meter, and the tortoise has moved 1/4 meter. Achilles crosses the quarter-meter, and the tortoise has gone 1/8. Achilles jumps this 1/8, and the tortoise has gone 1/16, and so on. Zeno reasons that, since infinitely many events must occur for Achilles to catch the tortoise, Achilles can never catch the tortoise. (Achilles goes a meter, then another half, then a quarter, another 1/8th, ..., and each time, the tortoise is ahead, by less and less, of course). Zeno then drew the conclusion that, because logically Achilles can never catch the tortoise, and because we know he does, we have a contradiction. Thus, we conclude that the footrace was an illusion, and more generally, all movement is an illusion.

One could, of course, accept Zeno's reasoning, and believe the world is an illusion. An alternative, however, is to add up the distances Achilles moves before he catches the tortoise, and show that, just because there are infinitely many of them doesn't mean we can't add them up.

The "steps" we described are:

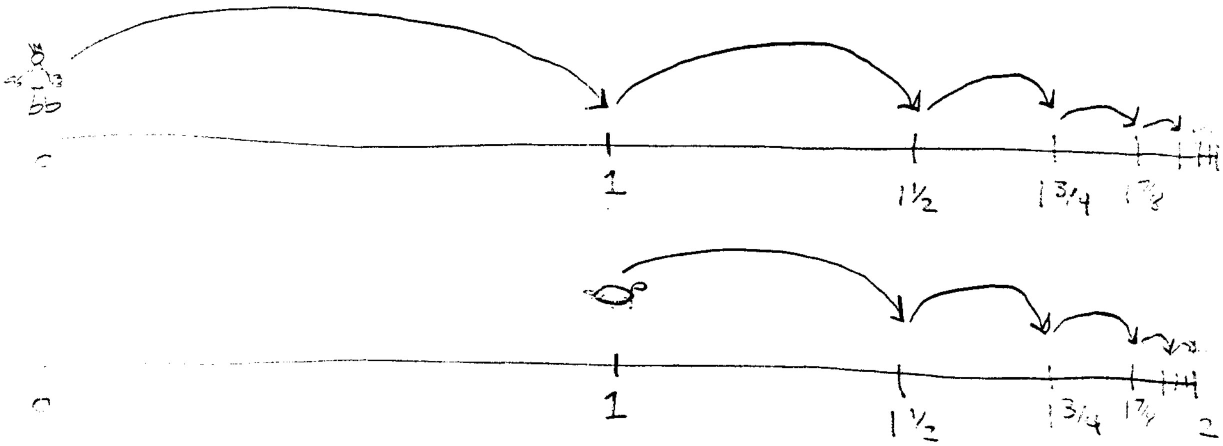

Figure 2.1 [illegible] $1 + \frac{1}{2} = 1\frac{1}{2}$, $1\frac{1}{2} + \frac{1}{4} = 1\frac{3}{4}$, etc., the tortoise [illegible]

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots = 2$$

To see these add up to 2, note that we may rewrite this as

$$(½)^0 + (½)^1 + (½)^2 + (½)^3 + (½)^4 + \ldots + (½)^n + \ldots$$

Further note

$$\sum_{i=0}^{n} (½)^i = (½)^0 + (½)^1 + \ldots + (½)^n = 2 - (½)^n$$

(The reader is requested to prove this by induction.)

Thus, an n grows toward infinity, $\sum_{i=0}^{n} (½)^n$ grows toward 2, reaching 2 as n reaches infinity. Thus, Achilles reaches the tortoise at the two-meter mark, and surges ahead of the tortoise after that.

This answer is consistent with the ordinary approach to this problem. Achilles runs twice as fast, and the tortoise has a 1-meter head start. The time the tortoise runs another meter is the time Achilles runs two meters, so they are tied at the two-meter mark, since 1 + 1 = 2 (head start + 1 meter for the tortoise = 2 meters for Achilles).

Clearly, there is no logical inconsistency as Zeno and many who followed him thought. Zeno couldn't imagine adding up infinitely many things. The purpose of this section is to show how such a process is accomplished. To generalize from this example, we showed that the sequence of numbers

$$1,\ 1\frac{1}{2},\ 1\frac{3}{4},\ 1\frac{7}{8},\ \ldots$$

gets ever closer to 2. That is, if we want it to be no more $\epsilon > 0$ away from 2, even when ϵ is very small (e.g. 0.00000000001), all we have to do is wait long enough, and the remaining terms of the sequence (those beyond, say, the $100{,}000{,}000{,}000^{th}$ term) will be within ϵ of 2. The point of this example (besides debunking Zeno) is that we can find "limits" of infinite sequences--infinite is within our understanding. In addition, such limits are defined by the sequence getting ever closer: as n gets very large, the n^{th} term of the sequence, and all those larger than n, get very close to the limit.

2.2 SEQUENCES

A sequence $\{x_n\}_{n=1}^{\infty}$, or just x_n, is an association of integers with real numbers, where x_n gives the n^{th} real number.

EXAMPLE:

$$x_n = (-1)^n$$

This sequence alternates -1, 1, -1, 1, ...

EXAMPLE:

$$x_n = 1/n$$

This sequence, 1, 1/2, 1/3, 1/4, 1/5, ... gets close to zero as n gets large. This suggests the notion of a limit.

DEFINITION 2.4: The sequence x_n converges to x_0 if

$$(\forall\ \varepsilon > 0)(\exists\ N_\varepsilon \in \mathbb{N}) \quad n \geq N_\varepsilon \Rightarrow |\ x_n - x_0\ | < \varepsilon$$

If so, we write $x_n \to x_0$ or $\lim_{n\to\infty} x_n = x_0$, and say x_0 is the limit of x_n.

The definition says, in words, that if we go out far enough ($n \geq N_\varepsilon$) we can make the terms as close to x_0 as we choose ($|x_n - x_0| < \varepsilon$). That is, no matter how close we want the terms x_n to get to x_0, but not equal, we can go out far enough in the sequence so that the remaining terms are that close.

The sequence $x_n = 1/n$ does indeed converge to zero. To check this, let $\varepsilon > 0$ be any number. Let N_ε be an integer larger than $1/\varepsilon$, that is

$$N_\varepsilon > 1/\varepsilon$$

Thus, if $n \geq N_\varepsilon$, $1/n \leq 1/N_\varepsilon < \varepsilon$. N_ε exists by (1.70).

thus, if $n \geq N_\varepsilon$,

$$|0 - x_n| = |0 - \frac{1}{n}| = \frac{1}{n} \leq \frac{1}{N_\varepsilon} < \varepsilon$$

Hence $1/n \to 0$.

The problem with this definition of convergence is that it requires one to guess the limit x_0 in order to check convergence. The following theorem avoids this problem.

DEFINITION 2.7: x_n is <u>cauchy</u> if $\forall \varepsilon > 0 \quad \exists N_\varepsilon \in N$

$m,n \geq N_\varepsilon \Rightarrow |x_n - x_m| < \varepsilon$

THEOREM 2.6: x_n converges if and only if x_n is cauchy.

We shall defer proof of this theorem briefly until developing one other result. The value of the theorem it that we may establish a limit point $x_0$exists without guessing the limit in advance, since the definition of a cauchy sequence does not involve x_0.

THEOREM 2.7 (Bolzano-Weierstrauss): Let $A \subseteq R$ contain infinitely many elements, and be bounded. Then

$(\exists a \in R)(\forall \varepsilon > 0)(\exists x \in A) \quad 0 < |x-a| < \varepsilon$

PROOF OF THEOREM 2.7: Let b be the bound of A, so $(\forall x \in A)$ $|x| \leq b$. Since A has infinitely many elements, there are infinitely many elements in [-b,0] or [0,b] or both. Choose the half with infinitely many elements (or choose one, if both), and divide it in half (e.g. if [0,b], then [0,b/2] and [b/2,b]). Now choose the quarter with infinitely many elements. Continue this process. If $[\alpha_i, \beta_i]$ is an interval with infinitely many elements; divide it into two intervals

$[\alpha_i, \frac{\alpha_i+\beta_i}{2}]$, $[\frac{\alpha_i+\beta_i}{2}, \beta_i]$ and choose the half with infinitely many elements.

This provides a series of intervals which we will denote $[\alpha_n,\beta_n]$ of length $b/2^{n-2}$ (if the first $[\alpha_1,\beta_1] = [-b,b]$ of length 2b). Further, α_n is an increasing sequence, and thus has a limit α_0, and β_n is a decreasing sequence, and has a limit of β_0. In addition,

$$\beta_n - \alpha_n = \frac{b}{2^{n-2}}$$

converges to zero, so $\beta_0 = \alpha_0$. This is our point a. Now, in any interval around a, that is $(a-\varepsilon, a+\varepsilon)$, there is an interval $[\alpha_n,\beta_n]$, if we choose n large enough so that

$$\frac{b}{2^{n-1}} < \varepsilon$$

In this inteval, there are infinitely many points of A. Choose any one to satisfy the theorem. Q.E.D.

The Bolzano-Weierstrauss Theorem proves the intuitively obvious fact that, if an interval has infinitely many points of A, then it is not possible to "space them out"; at least some must crowd together around a point.

PROOF OF THEOREM 2.6:

($\Rightarrow$) Suppose x_n converges to x_0. Let $\varepsilon > 0$. Then $\varepsilon/2 > 0$ and $\exists N_{\varepsilon/2}$ s.t.

$$n \geq N_{\varepsilon/2} \Rightarrow |x_n - x_0| < \varepsilon/2$$

Thus, if $n,m \geq N_{\varepsilon/2}$ (1.67)

$$|x_n - x_m| = |x_n - x_0 + x_0 - x_m| \leq |x_n - x_0| + |x_0 - x_m| =$$

$$|x_n - x_0| + |x_m - x_0| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

Thus $N_{\varepsilon/2}$ works for the cauchy definition.

(⇐) Now suppose x_n is cauchy

Case 1: $\{x_n | n \in N\}$ is finite.

Then we claim x_n is eventually constant.

Let $\varepsilon = \min \{|x_n - x_m| \ |x_n \neq x_m\}$. Since $\{x_n | n \in N\}$ is finite, $\varepsilon >$ 0. Since x_n is cauchy, $\exists N_\varepsilon$, $n,m \geq N_\varepsilon \Rightarrow |x_n - x_m| < \varepsilon$.

Thus $x_n = x_m$ for $n,m \geq N_\varepsilon$ (otherwise $|x_n - x_m| < |x_n - x_m|$ by the definition of ε). Thus, x_n trivially converges to x_{N_ε}.

Now suppose $\{x_n | n \in N\}$ is infinite. By the Bolzano-Weierstrauss theorem, $\exists\ x_0 \in R$ so that

$$0 < |x_j^\varepsilon - x_0| < \varepsilon$$

for some $x_j^\varepsilon \in \{x_n | n \in N\}$

Let $\varepsilon > 0$. $\exists N_{\varepsilon/2}$ s.t. $n,m \geq N_{\varepsilon/2}$ implies

$$|x_n - x_m| < \varepsilon/2$$

In addition, we can choose an x_j, $j > N_{\varepsilon/2}$, so that $0 < |x_j - x_0| <$ $\varepsilon/2$. Thus, if $n \geq N_{\varepsilon/2}$

$$|x_n - x_0| \leq |x_n - x_j + x_j - x_0| \leq |x_n - x_j| + |x_j - x_0| < \varepsilon/2 + \varepsilon/2 = \varepsilon \quad (1.67)$$

Therefore $x_n \to x_0$ Q.E.D.

Cauchy convergence is useful because it avoids having to know the limit to show a sequence converges. In particular, it allows us to establish the existence of a limit point. We shall see it in the study of fixed points to contraction mappings and again in the existence of solutions to differential equations.

The following theorems establish the existence of limits to some useful sequences.

THEOREM 2.8: If $x_n \to x_0$ and $y_n \to y_0$, then

(i) $(x_n + y_n) \to x_0 + y_0$

(ii) $x_n y_n \to x_0 y_0$

(iii) if $y_n \neq 0$ $\quad \frac{1}{y_n} \to \frac{1}{y_0}$

PROOF:

(i) $x_n \to x_0 \therefore \exists N_1 \quad n \geq N_1 \Rightarrow |x_n - x_0| < \varepsilon/2$

$y_n \to y_0 \therefore \exists N_2 \quad n \geq N_2 \Rightarrow |y_n - y_0| < \varepsilon/2$

Thus, if $n \geq N_\varepsilon = \max\{N_1, N_2\}$ (1.67)

$|x_n + y_n - (x_0 + y_0)| = |x_n - x_0 + y_n - y_0| \leq$

$|x_n - x_0| + |y_n - y_0| < \varepsilon/2 + \varepsilon/2 = \varepsilon$ Q.E.D.

(ii) $(\exists N_1)\ n \geq N_1 \Rightarrow |x_n - x_0| < \frac{1}{3} \min\{1, \frac{\varepsilon}{1+|y_0|}\}$

$(\exists N_2)\ n \geq N_2 \Rightarrow |y_n - y_0| < \frac{1}{3} \min\{1, \frac{\varepsilon}{1+|x_0|}\}$

if $n \geq \max\{N_1, N_2\}$ (1.67)

$|x_n y_n - x_0 y_0| = |(x_n - x_0)(y_n - y_0) + (x_n - x_0)y_0 + (y_n - y_0)x_0| \leq$

$|(x_n - x_0)(y_n - y_0)| + |y_0(x_n - x_0)| + |x_0(y_n - y_0)| =$

$|x_n - x_0|\ |y_n - y_0| + |y_0|\ |x_n - x_0| + |x_0|\ |y_n - y_0| <$

$$\left(\frac{\varepsilon}{3}\right)\left(\frac{1}{3}\right) + \frac{\varepsilon}{3}\,\frac{|y_0|}{1+|y_0|} + \frac{\varepsilon}{3}\,\frac{|x_0|}{1+|x_0|} \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

since $\frac{|y_0|}{1+|y_0|} < 1$, $\frac{|x_0|}{1+|x_0|} < 1$, $|x_n - x_0| < \frac{1}{3}$, $|x_n - x_0| < \frac{\varepsilon}{3}$

and $|y_n - y_0| < \frac{\varepsilon}{3}$.

(iii) $(\exists N_1)\ n \geq N_1 \Rightarrow |y_n - y_0| < \frac{\varepsilon|y_0|^2}{2}$

$(\exists N_2)\ n \geq N_2 \Rightarrow |y_n - y_0| < \frac{|y_0|}{2} \Rightarrow |y_n| > |\frac{y_0}{2}| \Rightarrow$

$$\frac{1}{|y_n|} < \frac{2}{|y_0|}$$

Thus, if $n \geq \max\{N_1, N_2\}$

$$|\frac{1}{y_n} - \frac{1}{y_0}| = |\frac{y_0 - y_n}{y_n y_0}| = \frac{|y_n - y_0|}{|y_n||y_0|} < \left(\frac{\varepsilon|y_0|^2}{2}\right)\left(\frac{2}{|y_0|}\right)\frac{1}{|y_0|} = \varepsilon$$

Q.E.D.

COROLLARY 2.9: if $y_n \to y_0 \neq 0$ and $x_n \to x_0$, then

$$\frac{x_n}{y_n} \to \frac{x_0}{y_0}$$

Corollary 2.9 follows from (ii) and (iii).

THEOREM 2.10: If $x_n \to x_0$ and $y_n \to x_0$ and $\forall n \in N \quad x_n \le z_n \le y_n$, then $z_n \to x_0$

PROOF: Note $|z_n - x_0| \le \max\{|x_n - x_0|, |y_n - x_0|\}$

Let $\varepsilon > 0$. $\exists N_1 \quad n \ge N_1 \Rightarrow |x_n - x_0| < \varepsilon$

$\exists N_2 \quad n \ge N_2 \Rightarrow |y_n - y_0| < \varepsilon$

Thus, if $n \ge \max\{N_1, N_2\}$

$|z_n - x_0| \le \max\{|x_n - x_0|, |y_n - y_0|\} < \max\{\varepsilon, \varepsilon\} = \varepsilon$

THEOREM 2.11: Suppose $A \subseteq R$ is bounded. Then $\exists$ a sequence x_n, $x_n \in A$, $x_n \to \sup A$.

PROOF: CASE 1: $\sup A \in A$. Then $x_n = \sup A$ for all n converges to $\sup A$.

CASE 2: $\sup A \notin A$. Let $x_1 \in A$. Then since $\frac{1}{2}(x_1 + \sup A) < \sup A$, $\exists\, x_2 \in A$, $\frac{1}{2}(x_1 + \sup A) < x_2 < \sup A$ (since $\frac{1}{2}(x_1 + \sup A)$ is not an upper bound)

In general, we define the sequence $x_n \in A$ by

$$\tfrac{1}{2}(x_{n-1} + \sup A) < x_n < \sup A$$

(since $\frac{1}{2}(x_{n-1} + \sup A)$ is not an upper bound, $\exists x_n \in A$ with the desired property);

it remains to be shown that $x_n \to \sup A$

$$\sup A \ge x_n > \tfrac{1}{2}(x_{n-1} + \sup A) > \tfrac{1}{2}\sup A + \tfrac{1}{2}(\tfrac{1}{2}(x_{n-2} + \sup A)) > \ldots$$

$$\tfrac{1}{2}\sup A + \tfrac{1}{4}\sup A + \ldots + \frac{1}{2^{n-1}}\sup A + \frac{1}{2^{n-1}} x_n =$$

$$(1 - \frac{1}{2^{n-1}})\sup A + \frac{1}{2^{n-1}} x_1 \to \sup A.$$

Thus, by Theorem 2.10, $x_n \to \sup A$ as desired Q.E.D.

2.3 CONTINUITY

The notion of continuity should capture the idea on no breaks in a curve. Figure 2a is continuous, while 2b has a break in the curve at x_0. What occurs there is that f(x) takes a large jump even as x takes a very small step.

DEFINITION 2.6: $f: R \to R$ is continuous at x_0 if

$\forall \varepsilon > 0 \quad \exists \delta > 0 \ |x-x_0| < \delta \Rightarrow |f(x)-f(x_0)| < \varepsilon$

if f is continuous at all x_0, f is said to be continuous.

Effectively, this definition says that, if x is close to x_0, ($|x-x_0| < \delta$) then f(x) gets close to $f(x_0)(|f(x) - f(x_0)|) < \varepsilon$. Thus, it captures the intuitive notion of no breaks or jumps, since these are points where f changes a lot even while x is altered only a tiny amount. The next theorem relates continuity to the behavior of sequences. Suppose $x_n \to x_o$ is a sequence. We may define a new sequence $y_n = f(x_n)$. For example, if $x_n = 1/n$, which converges to zero and $f(x) = x^2$, then $y_n = f(x_n) = xn^2 = 1/n^2$. Theorem 2.12 shows that, f is continuous at x_o if and only if $f(x_n) \to f(x_o)$ whenever $x_n \to x_o$. Thus, continuity lets us take limits inside a function:

$$F(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n)$$

since both are $f(x_0)$. That is, continuous functions commute with limits: it does not matter what order they are written in.

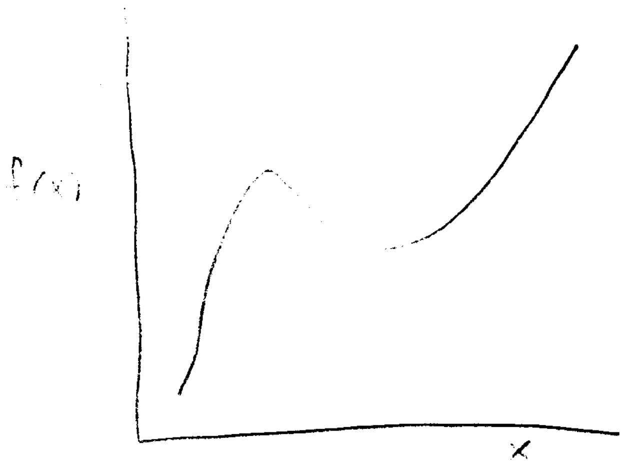

Figure 2.2a

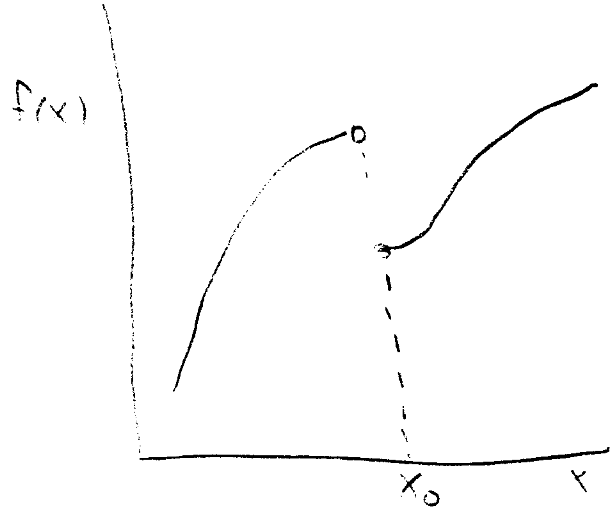

Figure 2.2b

THEOREM 2.12: f is continuous at x_0 if and only if for all sequences $x_n \to x_0$, $f(x_n) \to f(x_0)$

PROOF: ($\Rightarrow$) Let $\varepsilon > 0 \Rightarrow \exists\delta > 0$

$$|x-x_0| < \delta \Rightarrow |f(x)-f(x_0)| < \varepsilon$$

Now suppose $x_n \to x_0$. We must show $f(x_n) \to f(x_0)$.

Since $x_n \to x_0$, $\exists N_\delta$, $n \geq N_\delta \Rightarrow |x_n-x_0| < \delta$.

By continuity, this implies

$$n \geq N_\delta \Rightarrow |f(x_n)-f(x_0)| < \varepsilon$$

That is, $f(x_n) \to f(x_0)$ as desired.

($\Leftarrow$) by contrapositive: Suppose f is not continuous at x_0. Note that the negation of continuity is

$\sim\forall\varepsilon > 0 \quad \exists\delta > 0 \; (|x-x_0| < \delta \Rightarrow |f(x)-f(x_0)| < \varepsilon) \Leftrightarrow$

$\exists\varepsilon > 0 \quad \forall\delta > 0 \; (|x-x_0| < \delta \wedge |f(x)-f(x_0)| \geq \varepsilon)$.

This defines a sequence. Let x_1 satisfy

$$|x_1-x_0| < 1 \wedge |f(x_1)-f(x_0)| \geq \varepsilon$$

now choose x_2 so that

$$|x_2-x_0| < ½ \, |x_1-x_0| \wedge |f(x_2)-f(x_0)| \geq \varepsilon$$

i.e. choice of $\delta_2 = ½ \, |x_1-x_0|$

In general, let $\delta_n = ½ \, |x_{n-1}-x_{n-2}|$

Then

$$|x_n-x_0| < ½ \, |x_{n-2}-x_{n-2}| < ½\cdot½ \, |x_{n-2}-x_{n-3}| < \ldots < \frac{1}{2^{n-1}} |x_1-x_0|$$

Thus $|x_n - x_0| \to 0$ or $x_n \to x_0$

clearly $f(x_n)$ does not converge to $f(x_0)$, since

$$|f(x_n)-f(x_0)| \geq \varepsilon > 0 \qquad \text{Q.E.D.}$$

From Theorem 2.8 and Theorem 2.12, we obtain:

THEOREM 2.13: if f, g are continuous at x_0, then

(i) $f(x) + g(x)$ is continuous at x_0

(ii) $f(x)g(x)$ is continuous at x_0

(iii) if $g(x_0) \neq 0$ $f(x)/g(x)$ is continuous at x_0.

THEOREM 2.14: if g is continuous at x_0, and f is continuous at $g(x_0)$, then $f(g(x))$ is continuous at x_0.

PROOF: Let $\varepsilon > 0$

$$\exists\delta_1 > 0 \quad |y-g(x_0)| < \delta_1 \Rightarrow |f(y)-f(g(x_0))| < \varepsilon$$

$$\exists\delta_2 > 0 \quad |x-x_0| < \delta_2 \Rightarrow |g(x)-g(x_0)| < \delta_1$$

Thus, if $|x-x_0| < \delta_2$, then

$|g(x)-g(x_0)| < \delta_1$ and hence $(y = g(x))$

$$|f(g(x))-f(g(x_0))| < \varepsilon. \qquad \text{Q.E.D.}$$

THEOREM 2.15: The polynomials $\sum_{i=0}^{n} a_i x^i$ are continuous.

PROOF: By induction, $\sum_{i=0}^{0} a_i x^i = a_0$ is constant and hence continuous (any δ works).

$f(x) = x$ is continuous ($\delta = \varepsilon$ works).

Inductively, by part (ii) of Theorem 2.12, $a_i x^i$ is continuous. Now, using (i) in an induction, $\sum_{i=0}^{n} a_i x^i$ is continuous. Q.E.D.

THEOREM 2.16 (Intermediate Value Theorem): Suppose f is continuous, $a < b$, and $f(a) < 0 < f(b)$. Then $\exists x$, $a < x < b$, and $f(x) = 0$.

PROOF: Let $A = \{y | a \leq y \leq b \text{ and } f(y) < 0\}$. This is nonempty since $a \in A$, bounded by b. Hence $y_0 = \sup A$ exists. Further $\exists$ sequence $y_n \to y_0$, $y_n \in A$, by Theorem 2.11. If $f(y_0) = 0$, we're done. So suppose $f(y_0) < 0$. Then $f(y_0) < 0 < f(y_0+\varepsilon)$ for all $\varepsilon > 0$ (since, if $f(y_0+\varepsilon) < 0$, $y_0 + \varepsilon \in A$ contradicting $y_0 = \sup A$), which contradicts continuity.

COROLLARY to the intermediate value theorem: Suppose $f:[a,b] \to [a,b]$ and $f(a) > a$, $f(b) < b$. Then $\exists x$, $a < x < b$, $f(x) = x$. Such an x is called a fixed point.

To prove this corollary, merely note that $g(x) = f(x) - x$ satisfies the hypothesis of Theorem 2.16.

2.4 DERIVATIVES

DEFINITION 2.7: a is the derivative of f at x_0 if

$$\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = a$$

If f has a derivative at x_0, we will denote it by $f'(x_0)$.
If f has a derivative at all x, we say f is differentiable.

Note that $f'(x_0)$ is the slope of f at a point x_0. $f(x)-f(x_0)$ is the change in f when x changes by $x-x_0$, and thus

$$\frac{f(x)-f(x_0)}{x-x_0} = \frac{\Delta f}{\Delta x}$$

is the slope of f (see figure 2.3)

As we send x toward x_0, $(f(x)-f(x_0))/(x-x_0)$ more closely approximates the slope of f at x_0. In the limit, we obtain exactly the slope of f.

Consider figure 2.4. f is not differentiable at x_0, because the slope from the left does not equal the slope from the right. Thus, the limit will not exist in general. To see this formally, note that the sequence

$$x_n = x_0 + (-1)^n \, 1/n$$

converges to x_0, but

$$\frac{f(x_n)-f(x_0)}{x_n-x_0}$$

merely alternates between the two slopes, and hence does not converge.

Any differentiable function is continuous, as we see from the next theorem.

THEOREM 2.17: If f is differentiable, then f is continuous.

PROOF: Rewriting the definition of differentiability, we have

$$\lim_{x \to x_0} \frac{f(x)-f(x_0)-f'(x_0)(x-x_0)}{|x-x_0|} = 0$$

or $\forall \varepsilon \; \exists \delta \quad |x-x_0| < \delta \Rightarrow$

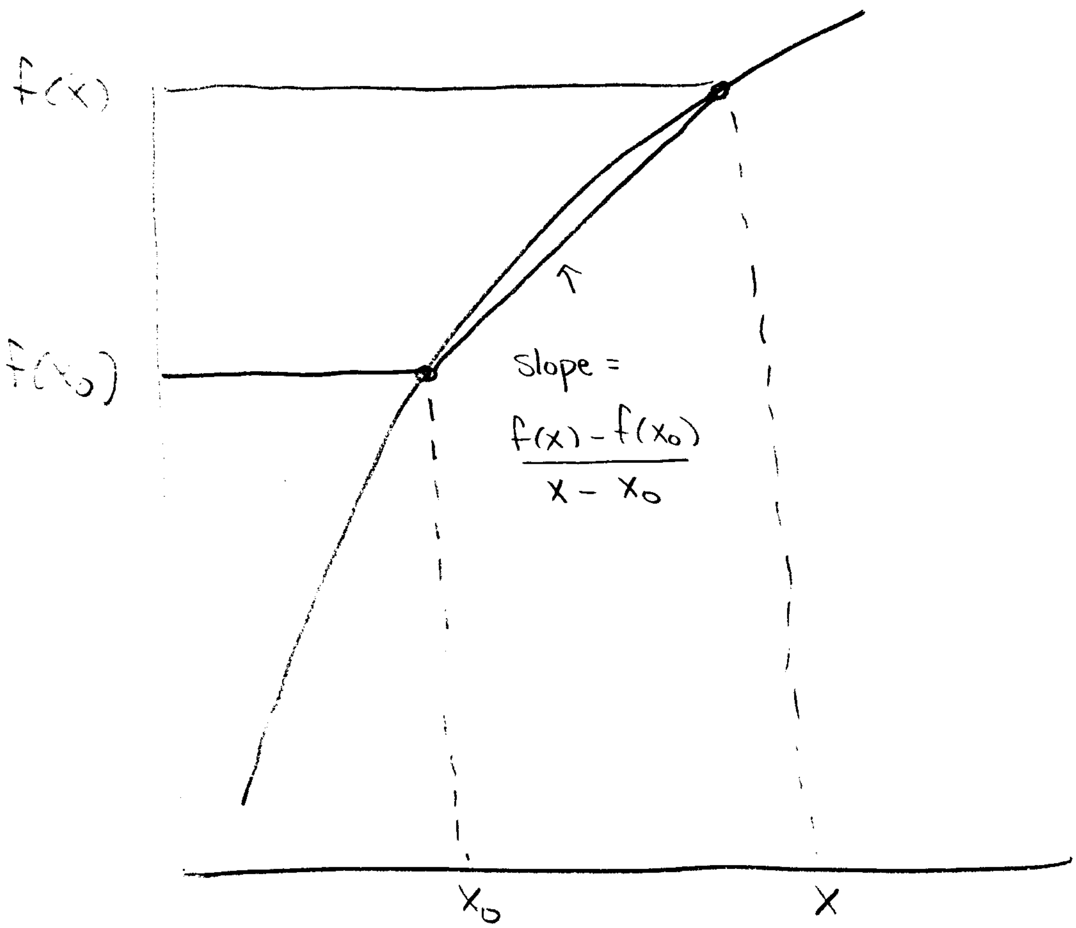

figure 2.3

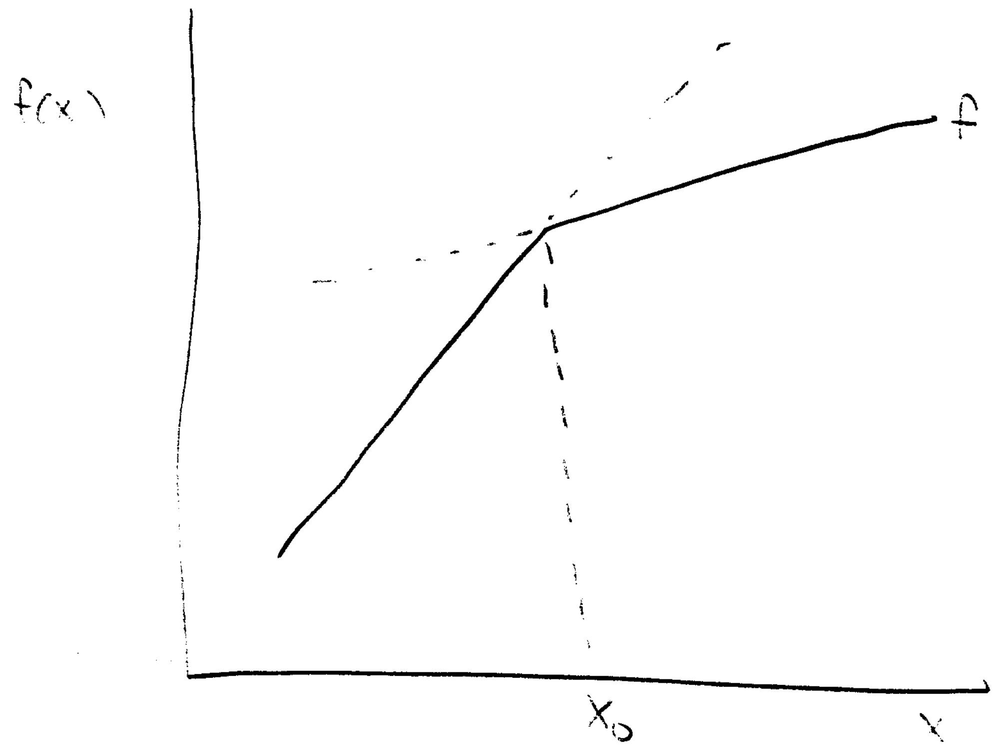

figure 2.4

$$\frac{|f(x)-f(x_0)-f'(x_0)(x-x_0)|}{|x-x_0|} < \varepsilon$$

or, equivalently

$$|f(x)-f(x_0)-f'(x_0)(x-x_0)| < \varepsilon\ |x-x_0|$$

Let $\hat{\delta} = \min \{\delta, \frac{\varepsilon}{\varepsilon+|f'(x_0)|}\}$

Then, if $|x-x_0| < \hat{\delta}$

$$|f(x)-f(x_0)| \le |f(x)-f(x_0)-f'(x_0)(x-x_0)| + |f'(x_0)(x-x_0)| <$$

$$\varepsilon|x-x_0| + |f'(x_0)|\ |x-x_0| =$$

$$(\varepsilon + |f'(x_0)|)\ |x-x_0| < \varepsilon$$

Q.E.D.

Another notation for the derivative is

$$\frac{d}{dx} f(x) = f'(x)$$

This notation is reminiscent of the slope $\Delta f/\Delta x$, and df/dx is merely the limit of the slope as $\Delta x \to 0$. This notation makes it easier to state the following theorem:

THEOREM 2.18: if f and g are differentiable, then

(i) $\frac{d}{dx} (f(x) + g(x)) = f'(x) + g'(x)$

(ii) $\frac{d}{dx} f(x)g(x) = f'(x)g(x) + f(x)g'(x)$

(iii) $\frac{d}{dx} f(g(x)) = f'g(x))g'(x)$

PROOF: (i) Let $\varepsilon > 0$. Since f is differentiable at x_0,

$$\exists \delta_1 > 0 \text{ s.t. } |x-x_0| < \delta_1 \Rightarrow \frac{|f(x)-f(x_0)-f'(x_0)(x-x_0)|}{|x-x_0|} < \varepsilon/2$$

Similarly $\exists \delta_2 > 0$ s.t. $|x-x_0| < \delta_2 \Rightarrow$

$$\frac{|g(x)-g(x_0)-g'(x_0)(x-x_0)|}{|x-x_0|} < \varepsilon/2$$

Thus if $|x-x_0| < \delta = \min\{\delta_1, \delta_2\}$

$$\frac{|f(x)+g(x)-(f(x_0)+g(x_0)-(f'(x_0)+g'(x_0))(x-x_0)|}{|x-x_0|} =$$

$$\frac{|f(x)-f(x_0)-f'(x_0)(x-x_0)+g(x)-g(x_0)-g'(x_0)(x-x_0)|}{|x-x_0|} \leq$$

$$\frac{|f(x)-f(x_0)-f'(x_0)(x-x_0)|+|g(x)-g(x_0)-g'(x_0)(x-x_0)|}{|x-x_0|} <$$

$\frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Therefore

$$\lim_{x \to x_0} \frac{f(x)+g(x)-f(x_0)+g(x_0)}{x-x_0} = f'(x_0)+g'(x_0).$$

(ii) Note

$$f(x)g(x)-f(x_0)g(x_0)-[f'(x_0)g(x_0)+g'(x_0)f(x_0)](x-x_0) =$$

$$g(x_0)[f(x)-f(x_0)-f'(x_0)(x-x_0)]+f(x_0)[g(x)-g(x_0)-g'(x_0)(x-x_0)]$$

$$+f(x)g(x)-f(x)g(x_0)-f(x_0)g(x)+f(x_0)g(x_0)$$

$$= g(x_0)[f(x)-f(x_0)-f'(x_0)(x-x_0)] + f(x_0)[g(x)-g(x_0)-g'(x_0)(x-x_0)]$$

$$+[f(x)-f(x_0)][g(x)-g(x_0)]$$

Thus

$$\lim_{x \to x_0} \frac{|f(x)g(x)-f(x_0)g(x_0)-(f'(x_0)g(x_0)+f(x_0)g'(x_0))(x-x_0)|}{|x-x_0|} \leq$$

$$\lim_{x \to x_0} \frac{|g(x_0)||f(x)-f(x_0)-f'(x_0)(x-x_0)|}{|x-x_0|}$$

$$+ \frac{|(f(x_0)||g(x)-g(x_0)-g'(x_0)(x-x_0)|}{|x-x_0|}$$

$$+ \frac{|f(x)-f(x_0)||g(x)-g(x_0)|}{|x-x_0|}$$

The first two terms go to zero by differentiability.

The third term goes to zero since

$$\lim_{x \to x_0} \frac{|f(x)-f(x_0)||g(x)-g(x_0)|}{|x-x_0|}$$

$$= \lim_{x \to x_0} \frac{|f(x)-f(x_0)|}{|x-x_0|} \lim_{x \to x_0} |g(x)-g(x_0)|$$

$$= |f'(x_0)| \lim_{x \to x_0} |g(x)-g(x_0)| = 0 \text{ by continuity.}$$

(iii)

$$\lim_{x \to x_0} \frac{f(g(x))-f(g(x_0))}{x-x_0} =$$

$$\lim_{x \to x_0} \frac{f(g(x))-f(g(x_0))}{g(x)-g(x_0)} \cdot \frac{g(x)-g(x_0)}{x-x_0} =$$

$$\lim_{x \to x_0} \frac{f(g(x))-f(g(x_0))}{g(x)-g(x_0)} \lim_{x \to x_0} \frac{g(x)-g(x_0)}{x-x_0} =$$

$$\left| \lim_{y \to y_0} \frac{f(y)-f(y_0)}{y-y_0} \right| g'(x_0) = f'(g(x_0))g'(x_0)$$

where $y = g(x)$, $y_0 = g(x_0)$, and the substitution of the limits follows from the continuity of g; since $x \to x_0$ implies $g(x) \to g(x_0)$. Q.E.D.

THEOREM 2.19: For integers $n \geq 0$, $\frac{d}{dx} x^n = nx^{n-1}$

PROOF: By induction.

$n = 0$: $\frac{d}{dx} x^0 = \frac{d}{dx} 1 = 0$. To prove this, let $f(x) = 1$ for all x.

$$\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{x \to x_0} \frac{1-1}{x-x_0} = \lim_{x \to x_0} 0 = 0, \text{ as desired.}$$

Now suppose $\frac{d}{dx} x^n = nx^{n-1}$.

$$\frac{d}{dx} x^{n+1} = \frac{d}{dx} x(x^n) = x^n + x \frac{d}{dx} x^n =$$

$x^n + x(nx^{n-1}) = (n+1)x^n$ as desired. However, we used $\frac{d}{dx} x = 1$ in the above. It remains to show this fact. If $f(x) = x$

$$\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{x \to x_0} \frac{x-x_0}{x-x_0} = \lim_{x \to x_0} 1 = 1$$

Q.E.D.

Inverse functions, written $f^{-1}(x)$, reverse the operation of f, so that $f(f^{-1}(x)) = x$.

THEOREM 2.20: If f^{-1} exists and f is differentiable, $f'(f^{-1}(x)) \neq 0$, then

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$

PROOF: $1 = \frac{d}{dx} x = \frac{d}{dx} f(f^{-1}(x)) = f'(f^{-1}(x))\, f^{-1\,'}(x)$. Q.E.D.

f^{-1} will exist when f does not change direction.

Note in the figure 2.5b, f changes direction and hence there are two points mapped into $f(x_1) = f(x_2)$. So either one is a candidate for $f^{-1}(y)$.

Although integrals will be fully explored in a later chapter, we shall require some use of them now. The reader may think of an integral as an "anti derivative", that is, the operation which cancels derivatives, as given by

THEOREM 2.21 (Fundamental Theorem of Calculus): If F is differentiable:

$$F(b) - F(a) = \int_a^b F'(x)\, dx$$

The proof is deferred. From Theorem 2.18 (ii), we have

$$\frac{d}{dx} u(x)\, v(x) = u(x)\, v'(x) + u'(x)\, v(x)$$

and this yields integration by parts:

$$\int_a^b u(x)v'(x)dx = \int_a^b \frac{d}{dx} u(x)v(x) - v(x)u'(x)dx$$

$$= \int_a^b \frac{d}{dx} u(x)v(x) - \int_a^b v(x)u'(x)dx$$

$$= u(b)v(b) - u(a)v(a) - \int_a^b u'(x)v(x)dx$$

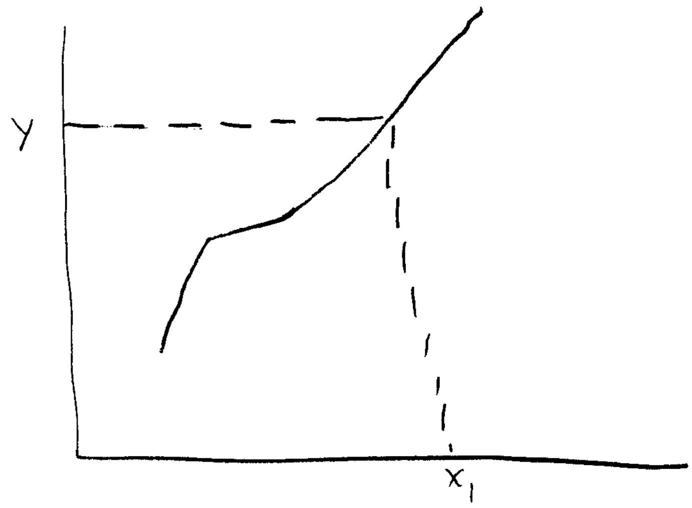

figure 2.5 a

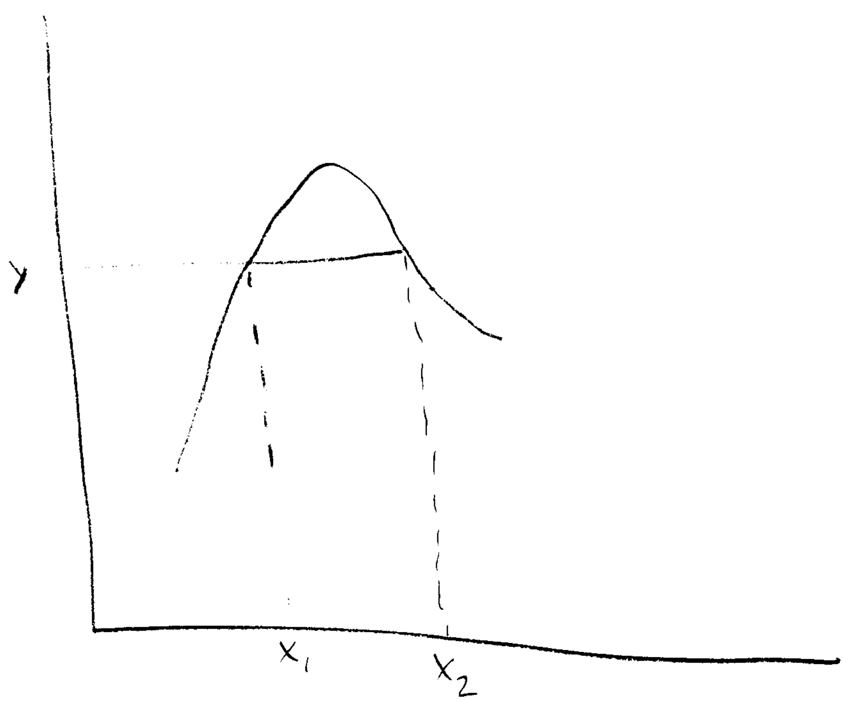

figure 2.5 b

Thus, we have

$$f(b) - f(a) = \int_a^b f'(x)dx$$

$$= f'(a)(b-a) + \int_a^b f''(x)(b-x)dx$$

(letting $v = -(b-x)$, $u = f'$, and noting $v(b) = 0$)

$$= f'(b)(b-a) + f''(x)\,\frac{(b-a)^2}{2} + \int_a^b f'''(x)\,\frac{(b-x)^2}{2}\,dx$$

(letting $v = -\frac{(b-x)^2}{2}$, $u = f''(x)$).

If we denote the n^{th} derivative of f by $f^{(n)}(x)$, we will by induction, have

$$f(b) = f(a) + f'(a)(b-a)+\ldots+\ f^{(n)}\,\frac{(a)(b-a)^n}{n!} + \int_a^b f^{(n+1)}(x)\,\frac{(b-x)^n}{n!}\,dx$$

If we continue this process indefinitely, we obtain the Taylor series expansion

$$f(b) = \sum_{n=0}^{\infty} f^{(n)}(a)\,\frac{(b-a)^n}{n!}$$

where $f^{(o)}(x) = f(x)$ and $0! = 1$.

This process clearly requires $f^{(n)}(x)$, the n^{th} derivative, to be always defined on [a,b]. In addition, the sequence

$$y_m = \sum_{n=0}^{m} f^{(n)}(a)\,\frac{(b-a)^n}{n!}$$

must converge to something for the Taylor series to be meaningful. If so, f is said to be analytic.

This process also allows local approximations.

THEOREM 2.22: (mean value theorem of integral calculus). If f is continuous, there is a $y \in [a,b]$ so that

$$\int_a^b f(x)dx = f(y)(b-a)$$

PROOF:

Let $f(y_0) = \sup\{f(x) \mid a \leq x \leq b\}$

$f(y_1) = \inf\{f(x) \mid a \leq x \leq b\}$

Then

$$f(y_0)(b-a) = \int_a^b f(y_0)dx \leq \int_a^b f(x)dx \leq \int_a^b f(y_1)dx = f(y_1)(b-a)$$

Thus

$g(y) = \int_a^b f(x)dx - f(y)(b-a)$ satisfies

$$g(y_1) > 0 > g(y_0)$$

By the mean value theorem, there is a y, between y_0 and y_1, with $g(y) = 0$. Q.E.D.

Since

$$f(b) = f(a) + f'(a)(b-a) + \int_a^b f''(x)(b-x)dx$$

a result similar to the above theorem shows $\exists\ y \in [a,b]$ so that

$$\int_a^b f''(x)(b-x)dx = \frac{1}{2}(b-a)^2 f''(y)$$

This provides the second order approximation

$$f(b) = f(a) + f'(a)(b-a) + \frac{1}{2}(b-a)^2 f''(y)$$

for some $y \in [a,b]$.

DEFINITION 2.8: f is concave if $\forall x, y, \lambda \in [0,1]$

$f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y)$.

Concavity is illustrated in figure 2.6.

$\lambda x + (1-\lambda)y$ is a point somewhere between x and y on the line segment connecting $(x, f(x))$ and $(y, f(y))$ in the plane, concavity says the function lies above this line segment, i.e. $f(\lambda x + (1-\lambda)y)$ exceeds $\lambda f(x) + (1-\lambda)f(y)$.

Consider a person facing a gamble where they get \$x with probability λ and \$y with probability $(1-\lambda)$. The average payout of this gamble is λ\$x + $(1-\lambda)$\$y. If f is the person's value of money, then concavity means that the person prefers the average payout to the gamble, that is, the value (or utility) of the average payout exceeds the average value from the gamble $(\lambda f(x) + (1-\lambda) f(y))$. Such a person is said to be risk averse.

THEOREM 2.23: If f'' is continuous, then the following are equivalent

(i) f is concave

(ii) $f(x) \leq f(y) + f'(y)(x-y)$

(iii) $f''(x) \leq 0$

PROOF:

(i) $\Rightarrow$ (ii): LET $g(\alpha) = f(\alpha x + (1-\alpha)y) - (\alpha f(x) + (1-\alpha)f(y))$

$g(o) = f(y) - f(y) = 0$.

By concavity, $g(\alpha) \geq 0$, so $g'(0) \geq 0$. Therefore $0 \leq g'(0) =$

$f'(y)(x-y) - (f(x) - f(y))$

or

$f(x) \leq f(y) + f'(y)(x-y)$

(ii) $\Rightarrow$ (iii) From our approximation result, $\exists\ z \in [x,y]$ so that

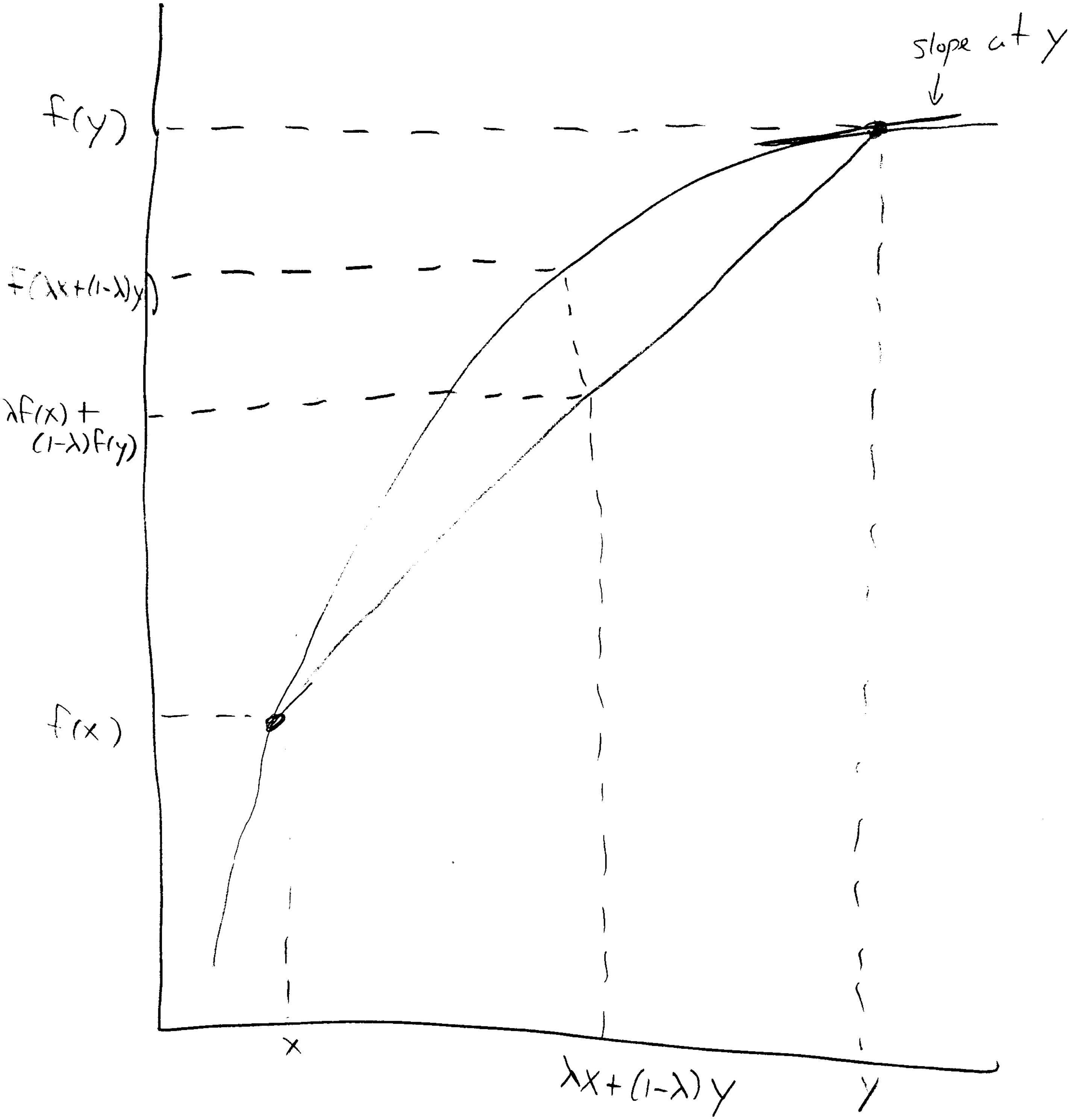

figure 2.6

$$f(x) = f(y) + f'(y)(x-y) + \frac{1}{2}(x-y)^2 f''(z)$$

but $f(x) - f(y) - f'(y)(x-y) \le 0$ by (ii), so

$$\frac{1}{2}(x-y)^2 f''(z) \le 0$$

and thus $f''(z) \le 0$

sending $x \to y$ gives $f''(y) \le 0$ since $x \le z \le y$ sends z to y, by Theorem 2.10.

(iii) $\Rightarrow$ (ii): if $f'' \le 0$, again using the approximation,

$$f(x) = f(y) + f'(y)(x-y) + \frac{1}{2}(x-y)^2 f''(z) \le f(y) + f'(y)(x-y)$$

since a non positive term has been dropped.

(ii) $\Rightarrow$ (i). LET $z = \lambda x + (1-\lambda)y$.

Then $x - z = (1-\lambda)(x-y)$

$y - z = \lambda(y-x) = -\lambda(x-y)$

from (ii):

$$f(x) \le f(z) + f'(z)(x-z) = f(z) + (1-\lambda)f'(z)(x-y)$$

$$f(y) \le f(z) + f'(z)(y-z) = f(z) - \lambda f'(z)(x-y)$$

so $\lambda f(x) \le \lambda f(z) + \lambda(1-\lambda)f'(z)(x-y)$, and

$$(1-\lambda)f(y) \le (1-\lambda)f(z) - \lambda(1-\lambda)f'(z)(x-y)$$

summing: $\lambda f(x) + (1-\lambda)f(y) \le f(z) = f(\lambda x + (1-\lambda)y)$

NOTE: (i) $\Rightarrow$ (iii) since (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii)

and (iii) $\Rightarrow$ (i) since (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i). Q.E.D.

This theorem provides three different characterizations of concavity. Part (iii) says the slope of f is non increasing, as $\frac{d}{dx} f'(x) \le 0$.

Part (ii) says, for $x < y$

$$\frac{f(x) - f(y)}{x - y} \geq f'(y)$$

so that the slope from x to y (x < y) is greater than the slope at y (see figure 2.6).

DEFINITION 2.9: f is convex if $\forall$ x, y, $\lambda \in [0,1]$

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

THEOREM 2.24: The following are equivalent

(i) f is convex

(ii) -f is concave

(iii) $f(x) \geq f(y) + f'(y)(x-y)$

(iv) $f''(x) \geq 0$

The proof is straightforward from the definitions and the previous theorem.

The final topic of this section is L'Hôpital's rule.

THEOREM 2.25: Suppose $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0$, and f and g are differentiable.

Then $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$

SKETCH OF PROOF: For x very close to x_0, f(x) is approximately $f(x_0) + f'(x_0)(x-x_0) = f'(x_0)(x-x_0)$, and similarly for g(x). Thus

$$\frac{f(x)}{g(x)} \text{ is approximately } \frac{f(x_0) + f'(x_0)(x-x_0)}{g(x_0) + g'(x_0)(x-x_0)} = \frac{f'(x_0)}{g'(x_0)} .$$

This argument applies if $\frac{f'(x)}{g'(x)}$ is continuous, i.e. $\lim_{x \to x_0} \frac{f'(x)}{g'(x)} = \frac{f'(x_0)}{g'(x_0)}$.

APPLICATIONS:

$$\lim_{x \to 0} x \log x = \lim_{x \to 0} \frac{\log x}{1/x} = \lim_{x \to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \to 0} - x = 0$$

$$\lim_{x \to 0} \frac{1}{x} (1-e^{-\lambda x}) = \lim_{x \to 0} \frac{\lambda e^{-\lambda x}}{1} = \lambda.$$

2.5 THE EXPONENTIAL FUNCTION

The exponential function is defined by

$$e^x = \lim_{n \to \infty} \sum_{i=0}^{n} \frac{x^i}{i!} = \sum_{i=0}^{\infty} \frac{x^i}{i!} \quad (1)$$

It can be shown that $\sum_{i=0}^{n} \frac{x^i}{i!}$ converges for all x. Immediately

$$e^0 = 1 \quad (2)$$

Also

$$e^x e^y = e^{x+y} \quad (3)$$

$$\frac{d}{dx} e^x = e^x \quad (4)$$

From (2) and (3)

$$e^{-x} = 1/e^x. \tag{5}$$

From (1) $e^x > 0$ for $x > 0$, and from (5), $e^x > 0$ for $x < 0$. Thus, e^x is positive. Using (4) inductively, all derivatives of e^x are positive. Thus e^x is convex (since second derivative is positive).

$$\lim_{x\to-\infty} e^x = 0$$

$$\lim_{x\to\infty} e^x = \infty$$

LET log x, sometimes written ln x, be the inverse function of e^x. log x is defined for $x > 0$, since this is the range of e^x. By Theorem 2.20:

$$\frac{d}{dx} \log x = 1/x$$

Thus, log x is an increasing concave function, and $\log 1 = 0$. These are graphed in figure 2.7.

$$e = e^1 = \sum_{n=0}^{\infty} \frac{1}{n!} = 2.718...$$

In addition, for scalars λ, α

$$(e^\alpha)^\lambda = e^{\lambda\alpha} = (e^\lambda)^\alpha$$

Thus, if $\lambda = \log x$,

$$(e^\alpha)^{\log x} = e^{\alpha \log x} = (e^{\log x})^\alpha = x^\alpha$$

or

$$x^\alpha = e^{\alpha \log x}$$

It follows, then, that

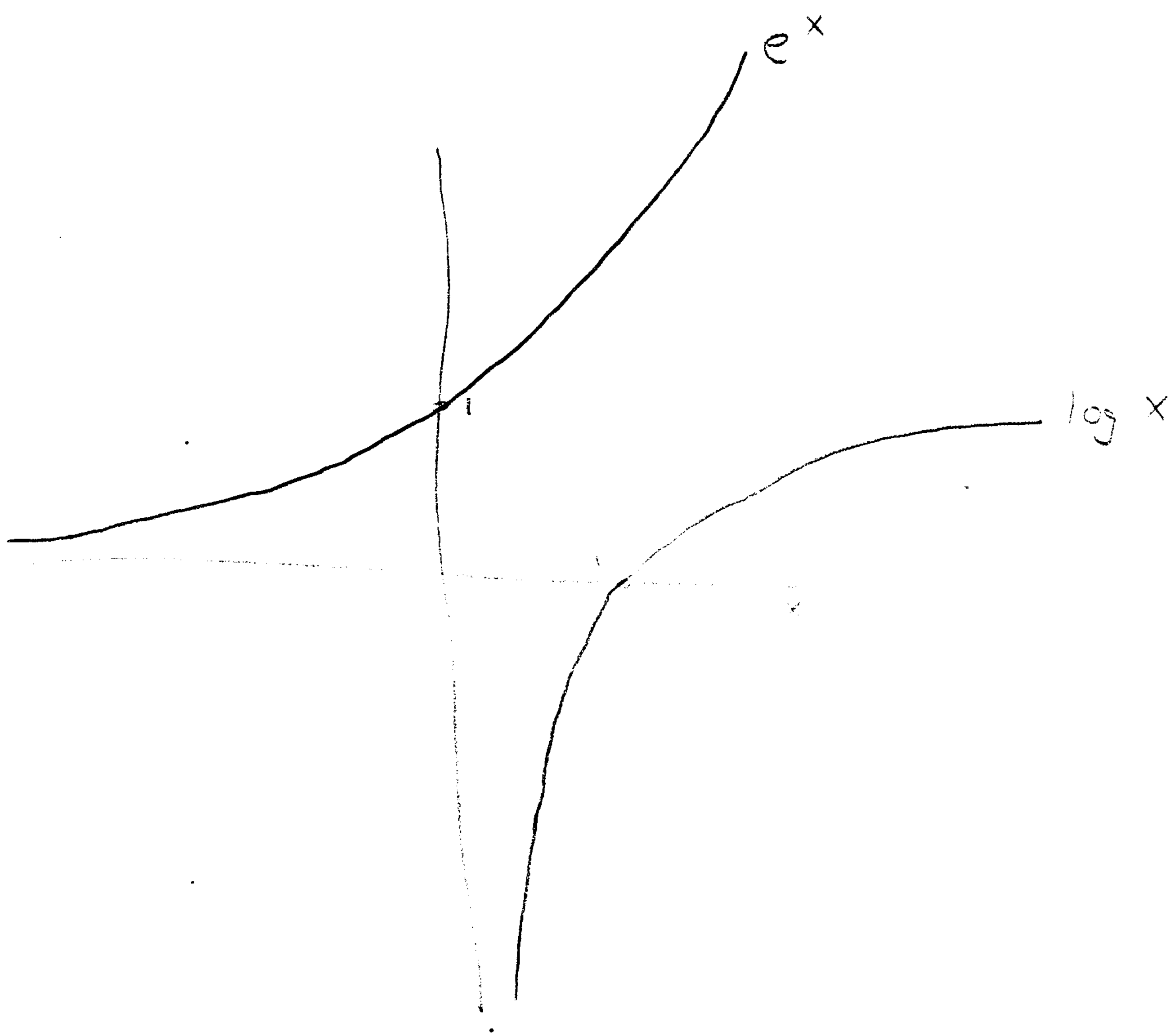

figure 2.17

$$\frac{d}{dx} x^{\alpha} = e^{\alpha \log x} (\alpha \frac{d}{dx} \log x) = x^{\alpha} \frac{\alpha}{x} = \alpha x^{\alpha-1}.$$

2.6 CONTRACTION MAPPINGS

DEFINITION 2.10: $f: R \to R$ is a contraction mapping if $\exists \lambda$, $0 \leq \lambda < 1$, $|f(x) - f(y)| \leq \lambda|x-y|$.

Intuitively, a contraction mapping squeezes the domain, since f(x) and f(y) are closer together than x and y.

THEOREM 2.26: If f is a contraction mapping, then $\exists! x^*$, $f(x^*) = x^*$.

PROOF: Choose x_0 arbitrarily, and define a sequence by $x_n = f(x_{n-1})$, n = 1,2,3.... We show first that x_n is cauchy, and hence has a limit, x^*. LET n > m. (1.67)

$|x_n - x_m| = |x_n - x_{n-1} + x_{n-1} - x_{n-2} + \ldots + x_{m+1} - x_m| \leq$ $|x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \ldots + |x_{m+1} - x_m| \leq (\lambda^{n-1} + \lambda^{n-2} + \ldots + \lambda^m)|x_1 - x_0|$.

Since $|x_k - x_{k-1}| = |f(x_{k-1}) - f(x_{k-2})| \leq \lambda|x_{k-1} - x_{k-2}|$, and by induction, $|x_k - x_{k-1}| \leq \lambda^{k-1}|x_1 - x_0|$. But $\lambda^{n-1} + \ldots + \lambda^m = (\lambda^m - \lambda^n)/(1-\lambda)$. Thus

$$|x_n - x_m| \leq ((\lambda^m - \lambda^n)/(1-\lambda))|x_1 - x_0| \leq$$

$$\frac{\lambda^m}{1-\lambda} |x_1 - x_0|$$

Thus, if we choose N_0 so that $\lambda^{N_0} < \frac{\varepsilon(1-\lambda)}{|x_1 - x_0|}$ we have, if $n,m \geq N_0$,

$$|x_n - x_m| \leq \frac{\lambda^m}{1-\lambda} |x_1 - x_0| \leq \frac{\lambda^{N_0}}{1-\lambda} |x_1 - x_0| < \varepsilon .$$

If $x_1 = x_0$, x_0 is our limit.

That is, x_n is cauchy, and has a limit x^*. Moreover, since f is continuous:

$$f(x^*) = f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = x^*$$

Finally, x^* is unique, for suppose

$f(x^*) = x^*$ and $f(y^*) = y^*$. Then

$|x^* - y^*| = |f(x^*) - f(y^*)| \leq \lambda\, |x^* - y^*|$.

If $|x^* - y^*| \neq 0$, then $1 \leq \lambda$, a contradiction. Thus $|x^* - y^*| = 0$, or $x^* = y^*$. Q.E.D.

If f is a contraction mapping, then there is exactly one solution to the equation $f(x) = x$. Such "fixed point" equations appear in calculating prices equating demand and supply. Suppose $c(q)$ is the industry marginal cost of supplying q, and $d(p)$ is the demand at price p. Then, at equilibrium, $p = c(q)$ and $q = d(p)$, that is $p = c(d(p))$. If $f(p) = c(d(p))$, then the existence of an equilibrium price reduces to the existence of such a fixed point to f.