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Amicable Divorce: Dissolving a Partnership with Simple Mechanisms

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This paper considers mechanisms for allocating the assets of a dissolving partnership that can be specified without information on the distribution of valuations of the asset or the level of risk aversion of the participants. For the case of a single asset and risk averse agents, three simple mechanisms are considered. These mechanisms can be ranked in terms of ex post efficiency and partially ranked in terms of interim expected utility. The efficiency of the "alternating selection" mechanism is considered for environments when there are many items to be allocated, agents are risk neutral, and transfers are prohibited. Conditions are given when a myopic strategy of picking the most preferred abailable item forms an equilibrium, and these conditions are satisfied under independence. Under the further assumption of identical distributions of valuations, it is shown that the total expected efficiency loss is at most $\frac{1}{2}$ of the maximal value of a single item, regardless of how many items are to be allocated. *Journal of Economic Literature* Classification Numbers: D39, D44, D52, D74, D81, D82.

1. INTRODUCTION

The typical dissolution of a partnership does not involve a business but rather a marriage. In both marriages and small partnerships, the assets involved typically compose most or all of the partners' assets. Thus, it is unlikely that the parties are risk neutral concerning the division of the assets of the partnership. In addition, the rules governing the division of the assets are typically either legislated or negotiated using lawyers and the courts, who are poorly informed about the partners' values of the assets. Consequently, it is unlikely that the courts will utilize any mechanism which is sensitive to the distribution of valuations or the level of risk aversion of the partners.

This paper will consider four simple mechanisms for allocating the asset or assets of a partnership among two partners. By simple mechanism,

[1] I thank two referees for suggestions, Jacques Robert for bringing the cake-cutting mechanism to my attention, and L. A. Shatford for anecdotal data.

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I mean mechanisms which can be specified without reference to either the distribution of valuations or the utility functions of the partners. There are several reasons for considering simple mechanisms over the solution to the ex ante utility maximization problem, which is the standard approach. For most environments, the sheer complexity of the mechanism design solution is a deterrent to its use.[2] Second, the sensitivity of the solution to the specification of the environment will deter people from employing the mechanism design solution, because people rarely know distributions of valuations, utility functions, and other characteristics of the environment with the degree of precision required to implement a mechanism design solution. Related to this point is the observation that institutions must operate in a variety of contexts, and the specification of the institution must be invariant to the details of the environment. In particular, courts are unlikely to solve nonlinear differential equations to allocate assets in a divorce, even presuming the court knew the utility functions of the parties. It is, on the other hand, quite reasonable to consider a court holding a standard auction. The analysis of simple mechanisms is important because economic agents will constrain themselves to these mechanisms in many circumstances.

Two obvious candidates for simple mechanisms to allocate a single good are first-price sealed bid autions and second-price sealed bid auctions, with the understanding that the winning bidder pays the other bidder, as opposed to paying a seller.[3] To force bids closer to valuations, it is useful to have the winning bidder pay *half* of the appropriate bid (either the winning bid in the case of the first-price auction or the losing bid in the case of a two-person second-price auction) to the losing bidder. To distinguish these mechanisms from standard first- and second-price auctions, I call these auctions the Winner's Bid Auction (WBA) and the Loser's Bid Auction (LBA), respectively. In the event that an *outside option*, the possibility of sale to a third party, exists, both the WBA and the LBA dictate sale to the third party if neither bidder's bid exceeds the third party's value, which for simplicity is taken to be known.

There are two ways to model an outside option. If the outside option is available to the parties individually, then the value of the outside option becomes the minimum possible value of each party and will tend to induce

[2] The central paper in this literature, Cramton, Gibbons, and Klemperer [1] (CGK), is an example of this, and will be discussed more extensively. Their paper is significantly more general than the present study with regard to allowing unequal shares in the partnership and more than two partners. They also propose a class of simple mechanisms to allocate efficiently. It turns out, however, that all but one member of this class results in an ex post inefficient allocation if a nontrivial outside option, not available to the parties individually, exists. They do not allow for risk aversion.

[3] Both of these mechanisms are special cases of CGK's $k+1$ *price auctions.*

a mass point in the value distribution at this point. This is the most plausible case and is quite similar to the case when no outside option exists. However, an outside option which the parties cannot exercise unilaterally might exist. Division of property often takes time, so that a currently available option may not be available by the time a complete settlement is reached. However, were both parties willing to agree now to exercise the outside option on a particular piece of property, then a quicker settlement may be available. In such a situation, the outside option value could exceed either party's valuation. We shall model the outside option in this manner, because it is a nearly free generality. However, the usual case is a trivial outside option, at or below the lower bound of the value distribution.

Under constant absolute risk aversion and symmetric independent private values, I show that both the WBA and the LBA possess symmetric bidding equilibria. The WBA is ex post efficient; that is, the good will be allocated to the agent who values it most highly. The LBA is ex post efficient only if no outside option exists and will allocate the good to one of the partners when it is valued more highly by an outsider with positive probability. If no outside option exists, low value bidders prefer the WBA and high value bidders prefer the LBA. Thus the mechanisms cannot be unambiguously ranked.

A third mechanism for allocating a single good among risk averse partners is the *cake-cutting mechanism*, or CCM, in which one party proposes a division and the other party chooses one of the parts of the division.[4] If the good itself is indivisible, the division can be accomplished with money, so that one party proposes a price and the other party chooses to either accept the price or to take the good and pay the first party the price. With symmetric information, the CCM produces an efficient allocation, and agents prefer to be the *proposer*, who suggests the split, rather than the *chooser*, who selects the side of the division he prefers. In the case of asymmetric information, the situation is dramatically different. The mechanism is ex post inefficient, and in an unusual way. If the proposer's value exceeds the median of the chooser's value distribution, the proposer may not obtain the item when he values it more than the chooser. If the proposer's value is less than the median, the proposer may obtain the item when he values it less than the chooser. If the outside option value is greater than the chooser's median value, the mechanism is efficient with respect to the outside option, but the inefficiency due to the proposer not

[4] Generalizations of this mechanism have been extensively studied. See Thompson and Varian [10] for a survey and Crawford and Heller [2] for a recent analysis. As far as I can ascertain, the mechanism has not been analyzed under asymmetric information in a Bayesian framework.

obtaining the item when he should remains. If the outside option value is less than the chooser's median value, the CCM is inefficient with respect to the outside option as well, and this option is not exercised in circumstances when it should be exercised. Moreover, in contrast to the full information case, under asymmetric information, the agents prefer to be the chooser.

Partners dissolving their partnership must often allocate many items that had similar purchase prices, such as books or compact discs. Arranging bids on such items would be quite time consuming if there are many of them. Consequently, the partners may use the *alternating selection mechanism*, or ASM, where one partner chooses an item, then the other chooses, and so forth. I address two questions about this mechanism under risk neutrality. First, when is a myopic strategy, where one selects the item most highly valued, optimal? The answer is roughly that the ordering of the items possessed by the agents is not negatively correlated, and that an agent does not learn anything about the other agent's ordering of the remaining items from the other agent's selection of one item. This is a very restrictive assumption, but it is satisfied if the agents' valuations for the items are independently distributed across items, although the valuations for a single item many be correlated across agents. The second question concerns the efficiency of this mechanism. Under the extreme assumption of *iid* valuations, the expected efficiency of the ASM is the expected maximum possible efficiency, minus at most $\frac{1}{2}$ the maximum value of a single item. This upper bound holds for any number of items to be allocated; that is, with a large number of items, the mechanism almost achieves the optimal allocation.

The paper is organized as follows. The second part of the paper considers the allocation of a single item under risk aversion. The analysis begins with the WBA, proceeds to the LBA, and then compares them in interim utility. The second part concludes with an analysis of the CCM. The third part of the paper focuses on the ASM under risk neutrality. All proofs are in the Appendix. Some of the deficiencies of the paper are discussed in the Conclusion.

2. Allocating a Single Item

Cramton, Gibbons, and Klemperer [1] (CGK) consider the allocation of a single item among partners under risk neutrality. The main focus of this paper is that the partnership allocation problem changes the individual rationality constraint, which says that each agent must be awarded an expected utility level at least as great as that which the agent would obtain in the absence of the mechanism. If each agent owns shares r_i in the item, CGK interpret the individual rationality constraint to ensure that each agent gets at least r_i of their value of the item. If the good is divisible, there is no problem implementing this default utility level. Under risk neutrality, the default utility level can also be implemented by giving each agent a probability r_i of obtaining the item, even if the good is indivisible. Such a random assignment will weaken the individual rationality constraint still further in the presence of risk aversion. CGK show that, if the shares are sufficiently symmetric, then there exists an efficient mechanism[5] for allocating the item that satisfies the individual rationality constraint. If the shares are sufficiently asymmetric, CGK show that there does not exist such an efficient mechanism.[6]

In this section, I will consider trying to achieve the CGK result with simple mechanisms that can be specified without reference to the distribution of valuations. I allow for constant absolute risk aversion, but follow CGK in assuming that valuations are identically and independently distributed. Because it is not obvious that one of the partners is necessarily the optimal owner of the item, I allow for an outside or third-party sale option.[7] For example, in a divorce, the house might go to one of the spouses or be sold on the open market and the cash divided among the divorcing partners.

2.1. *The Environment*

There are two agents,[8] 1 and 2, each with a private value, x_1 and x_2, respectively, for an object. The value x_i is known only to agent i. The values x_1 and x_2 are independently and identically distributed draws from a cumulative distribution function F, which has a continuous density f with support $[0, x_H]$. Both agents have constant absolute risk aversion with parameter λ, and I write the utility function as

$$u(y) = \lambda^{-1}[1 - e^{-\lambda y}], \tag{1}$$

where y is the net monetary value, either x_i minus payment made if the agent i obtains the good or the payment received if the agent does not obtain the good. The case of risk neutrality is obtained by taking the limit as $\lambda \to 0$.

There is an outside option, valued at $r \geqslant 0$. That is, the good can be sold

[5] Only mechanisms which do not lose money on average, hence do not require subsidies from outside, are considered.

[6] This generalizes Myerson and Satterthwaite [9].

[7] CGK show that, when no outside option exists, both the WBA and the LBA are allocatively efficient under risk neutrality. Their general result does not extend to the case with an outside option.

[8] Generalizations to the case of n agents are straightforward for the WBA and LBA but not, unfortunately, for the CCM.

for the known amount r to a third party.[9] In this case, both agents receive the monetary payment $\frac{1}{2}r$. It is useful to introduce a bid which has value r but means "sell to a third party." For the LBA and the CCM, it will be the case that bidders wish to bid the largest real number strictly less than r, and we let σ denote this fictitious bid. The alternative is to let a bid of r have different meanings with respect to exercising the outside option in different circumstances. One may think of σ as a bid infinitesimally below r. In all three mechanisms considered in this section, there will be a point of indifference where bidders switch from σ to a bid of at least r. It does not matter how this indifference is resolved to either agent.

There are two distinct notions of efficiency in this environment. The weaker notion is allocative, or ex post, efficiency, which occurs when the agent with the higher value obtains the good, provided that value exceeds r, and otherwise the outside option is exercised. If a mechanism has allocative efficiency, then there is no incentive to renegotiate the outcome by selling to the other agent or to a third party. The second, stronger notion requires optimal risk sharing as well as allocative efficiency, and I shall refer to this as ex ante efficiency. These two notions coincide in the case of risk neutrality. It is easily seen that ex ante efficiency requires equating the marginal utilities of the two agents, and thus the actual utility. That is, if $x_1 > \max\{r, x_2\}$, then agent 1 obtains the good and pays agent 2 the amount $\frac{1}{2}x_1$. It is trivially seen that ex ante efficiency is not incentive compatible. Consider an agent with value x, who reports a value y. His utility is

$$v(y, x) = \lambda^{-1}\left[1 - F(y)\, e^{-\lambda[x-(1/2)y]} - \int_y^{x_H} e^{-(1/2)s} f(s)\, ds\right],$$

which yields

$$\partial v(x, x)/\partial y = -e^{-(1/2)\lambda x} F(x) < 0.$$

Thus, faced with such a mechanism, an agent would choose to under-report his true value. This is intuitive, for the efficiency loss of such an action is negligible to the first order, while the reduction in payment

[9] If either agent can exercise the outside option individually, then no agent can have a value less than the outside option, and we set $r = 0$ to avoid a mass point in the distribution of values. However, it is plausible to think that an outside option might exist, but that immediate exercise of the option requires agreement of the parties, whereas individual exercise of the option requires delay until a full settlement, including agreement on other divisions not modeled here, is reached. In addition, liquidity issues often arise in the division of property, since a large amount of cash must be raised to pay lawyers. Therefore, the agreement to sell may have a higher value than the individual sale after agreement is reached, since other items will have to be sold at depressed prices.

provided one obtains the object is not. It is possible to achieve allocative efficiency, and indeed the winner's bid auction accomplishes it.

2.2. *The Winner's Bid Auction*

In the winner's bid auction, both bidders simultaneously submit sealed bids, which are opened simultaneously. The high bidder obtains the object and pays the loser $\frac{1}{2}$ of the high bid, provided this bid exceeds r, the outside option value. If the high bid does not exceed r, the object is sold to a third party and both bidders receive $\frac{1}{2}r$. Ties may be settled in any fashion, because they are probability zero events and are ignored by the agents.

LEMMA 1. *There is a symmetric equilibrium bidding function B_w, which is strictly increasing on $[r, x_H]$. $B_w(x) < x$, for $x > r$, and $B_w(r) = r$. B_w is given by*

$$B_w(x) = \begin{cases} \sigma & \text{for } x < r \\ \lambda^{-1} \log\left(F(x)^{-2}\left[e^{\lambda r}F(x)^2 + \int_r^x e^{\lambda y} 2F(y)f(y)\,dy\right]\right) & \\ & \text{for } x > r. \end{cases} \tag{2}$$

All proofs are given in the Appendix.[10]

The winner's bid auction achieves allocative efficiency, because the high bidder wins whenever his value exceeds r. There is a sense in which it is better to be a winner than a loser in the auction, in that a winner obtains a utility higher than that of a loser with the same value for the object, simply because the payoff to the winner, $x - \frac{1}{2}B_w(x)$, is greater than $\frac{1}{2}B_w(x)$, the payoff to the loser.

LEMMA 2. *The symmetric equilibrium bidding function, B_w, increases over $(r, x_H]$ as λ increases. For risk neutrality, and $x \geq r$,*

$$B_w(x)|_{\lambda=0} = F(x)^{-2}\left\{rF(r)^2 + \int_r^x y2F(y)f(y)\,dy\right\}$$

$$= x - F(x)^{-2}\int_r^x F(y)^2\,dy.$$

Finally, for $\lambda \geq 0$, the expected utility of an agent with value x strictly exceeds $u(\frac{1}{2}x)$, so that CGK individual rationality is satisfied.

[10] Proofs of Theorems 6 and 12 are long, tedious, and straightforward, and were provided to the referees, one of whom described them as suitable only for "mathematical masochists." They are available from the author on request.

Remark 1. Increases in the level of risk aversion tend to favor the losing bidder by driving bids up. The bidding function for risk neutral agents is precisely the bidding function that a bidder would use in a standard symmetric sealed bid auction[11] with reserve price r, provided there were *three* bidders and not two. While this observation is peculiar to the risk neutral case, and fails for $\lambda > 0$, it is still curious and counterintuitive that paying the other bidder half one's bid, rather than the seller, has the equivalent effect on the bidding function as introducing a third bidder in a standard auction.

The significance of the final part of Lemma 2 is that all types of bidders strictly prefer the WBA to getting half of the good. This shows that if the default, if either bidder refuses to participate, is to split the good equally, then all types of bidders prefer the WBA. Moreover, one can immediately deduce, by continuity, a result of CGK, that in a neighborhood of equal shares, there is an allocatively efficient mechanism which is strictly preferred by all types. This result is extended beyond that of CGK in two ways. First, it holds for constant absolute risk aversion and not just for risk neutrality. Because $u(\frac{1}{2}x) > \frac{1}{2}u(x)$, the bidders prefer the WBA to a 50% chance of getting the good as well. Second, there is a mechanism implementing this allocative efficiency which may be specified without reference to either the coefficient of risk aversion or the distribution of values, although of course it is assumed that the bidders themselves know these attributes of the environment. I now turn to the loser's bid auction.

2.3. *The Loser's Bid Auction*

The loser's bid auction is similar to the WBA, except the high bidder pays the low bidder the low bidder's bid, rather than the high bid. The third-party option bid, σ, is interpreted to have value r; that is, if the loser bids σ and the winner bids in excess of σ, then the winner pays the loser $\frac{1}{2}r$.[12] Thus, we can refer to $\max\{\sigma, y\}$ as σ if $y < r$, and y otherwise.

LEMMA 3. *There is a symmetric equilibrium bidding function B_1 for the LBA. $B_1(x_H) = x_H$, and for $x < x_H$, $B_1(x) > x$. $B_1(r) > r$. B_1 is strictly increasing, provided $B_1(x) > r$. Finally, the bidding function is given by*

$$B_1(x) = \max\left\{\sigma,\ -\lambda^{-1}\log\left((1-F(x))^{-2}\int_x^{x_H} e^{-\lambda y}2(1-F(y))f(y)\,dy\right)\right\}. \quad (3)$$

[11] By a standard sealed bid auction, I mean an auction in which the winning bidder pays a seller his bid, rather than paying the other bidder. See Milgrom and Weber [8] and McAfee and McMillan [5] for details and the bidding function.

[12] In the absence of the σ bid, a bidder may wish to bid r but never receive the item, which requires a bid less than r.

The LBA will pick the bidder with the higher value, but tends to be inefficient because a bidder with value r bids higher than r, in order to increase the amount he receives in the event that the other bidder wins. Thus, the outside option is not exercised in circumstances where it is efficient to do so.

LEMMA 4. *Provided $B_l(x) > r$, $B_l(x)$ is decreasing in λ. The limiting bidding function for risk neutrality is*

$$B_l(x)|_{\lambda=0} = \max\left\{\sigma,\ (1-F(x))^{-2}\int_x^{x_H} y2(1-F(y))f(y)\,dy\right\}.$$

Finally, for $\lambda \geqslant 0$ and $r = 0$, the expected utility of an agent with value x exceeds $u(\frac{1}{2}x)$.

In the case $r = 0$, the LBA produces an ex post efficient allocation. In this circumstance, it satisfies the CGK notion of individual rationality.

2.4 *Expected Utility in the WBA and LBA*

Because of the allocative inefficiency of the LBA, I shall only compare the two mechanisms when this allocative inefficiency does not arise, that is, when $r = 0$. This is not as unreasonable as it might at first appear. So long as either agent can *unilaterally* exercise the outside option, each agent's value of the object becomes the maximum of their own value and the value of the outside option, and no agent can have a value less than r. Of course, this tends to introduce a mass point in the distribution, which is assumed away for analytic convenience.

I adopt the notation $\pi_w(x)$ and $\pi_l(x)$ for the interim expected utilities[13] arising in equilibrium in the WBA and LBA, respectively.

THEOREM 5. *If $\lambda = 0$, $\pi_w(x) = \pi_l(x)$.*

Remark 2. Under risk neutrality, both auctions produce the same expected utilities, and there is no basis for choosing one over the other, provided no outside option exists.

THEOREM 6. *For $\lambda > 0$, $\pi_w(0) > \pi_l(0)$ and $\pi_w(x_H) < \pi_l(x_H)$. In addition, if B'_w and B'_l are always less than* 1, *then there exists a critical value x_c so that $\pi_w(x) < \pi_l(x)$ if and only if $x > x_c$.*

Remark 3. High value bidders prefer the LBA, while low value bidders prefer the WBA. The WBA presents high value bidders with the oppor-

[13] The interim expected utility is the expected utility after one knows his valuation x, but prior to bidding.

tunity to avoid risk by increasing their bids, making it more likely that the known bid is a winning bid. This tends to transfer money to the low value bidders. Similarly, low value bidders can increase the likelihood that their known bid is the losing bid by lowering their bid, which tends to transfer money to the high bidders. Thus, risk aversion works to make low value bidders prefer the WBA and high value bidders prefer the LBA.

Remark 4. The condition that B'_w and B'_l do not exceed 1 is reminiscent of a hazard rate condition. For the case of risk neutrality, $B'_w(x) < 1$ if and only if $\int_0^x (F(s)/F(x))^2\, ds$ is increasing in x. Similarly, for $\lambda = 0$, $B'_l(x) < 1$ if and only if $\int_x^{x_H} ((1-F(s))/(1-F(x)))^2\, ds$ is decreasing in x. These conditions are satisfied for the uniform distribution.

Because low value bidders prefer the WBA, high value bidders do not, and interim utility is increasing in value, the WBA tends to reduce the ex ante risk of the partners. Thus one might expect that the WBA is preferred ex ante, before valuations are known, to the LBA. The ex ante preference for the WBA remains an open question. For uniform F and small values of risk aversion (locally around zero), I have shown that the WBA is preferred ex ante to the LBA. This appears to be true for $\lambda < 50$, using graphics software.

2.5. *The Cake-Cutting Mechanism*

The cake-cutting mechanism, where one party proposes a division and the other party chooses which part of the division they want, has been extensively studied under full information (see Thompson and Varian [10]). The application of the mechanism to the allocation of an indivisible good involves one agent, the proposer, proposing a payment or transfer, and the other agent, the chooser, selecting either to receive the payment or to take the good and make the payment. As in the WBA and the LBA, it is useful to have the actual payment be half of the proposed payment. If b is the proposed payment, the chooser selects either $\frac{1}{2}b$ or the good, in which case the chooser pays the proposer $\frac{1}{2}b$.

Let x_p and x_c represent the values of the proposer and chooser, respectively. There are two significant properties of the full information case which do not carry over to the asymmetric case. Under full information, the solution is allocatively efficient, so that the proposer gets the good if $x_p > x_c$ and the chooser gets the good if $x_p < x_c$. Second, an agent prefers to be the proposer rather than the chooser. To see this, note that the proposer will make the chooser indifferent between the alternatives, so that $b = x_c$. If the proposer offers $b < x_c$, the chooser will take the good, as $x_c - \frac{1}{2}b > \frac{1}{2}b$, and the proposer obtains $\frac{1}{2}b$. Thus the proposer in this circumstance will raise b up to x_c. The case of $b > x_c$ is similar. By setting $b = x_c$, the proposer gets a choice of $x_p - \frac{1}{2}x_c$ and $\frac{1}{2}x_c$, and thus takes the good when $x_p > x_c$.[14] Thus, the proposer gets the maximum of $x_p - \frac{1}{2}x_c$ and $\frac{1}{2}x_c$, which exceeds $\frac{1}{2}x_p$, which is what the agent would obtain were he the chooser instead of the proposer.

The situation is quite different under asymmetric information. Let x_p and x_c be identically and independently distributed, with cumulative distribution function F. Let x_m be the median of F, that is, $F(x_m) = \frac{1}{2}$. I assume that F has a density f with interval support.

I introduce the outside option as follows. If the proposer offers a payment b in excess of r, then the outside option cannot be exercised. If the proposer offers σ, then the chooser may choose to exercise the outside option or not, but in either case, must pay the proposer $\frac{1}{2}r$. That is, a proposal of σ is a proposal to exercise the outside option. This means the chooser will select the outside option if and only if $x_c < r$, and the chooser is thus efficient with respect to the outside option.

If $b \geqslant r$, the chooser is offered a choice of x_c-$\frac{1}{2}b$ and $\frac{1}{2}b$ and thus chooses to take the good if $x_c > b$, which occurs with probability 1-$F(b)$. This leaves the proposer with

$$\begin{array}{ll} x_p - \frac{1}{2}b & \text{prob. } F(b) \\ \frac{1}{2}b & \text{prob. } 1 - F(b). \end{array}$$

Let u be the utility function of the proposer. There is no need to assume that the proposer displays constant absolute risk aversion, only that u is increasing and concave. I let $u(0) = 0$ without loss of generality. If the proposer has value x and suggests a payment $b \geqslant r$, he obtains expected utility

$$v(b, x) = u(x - \tfrac{1}{2}b)\, F(b) + u(\tfrac{1}{2}b)(1 - F(b)).$$

I assume that both of the standard hazard rate conditions, familiar in auction theory,[15] hold:

$$(\forall x) \qquad \frac{\partial}{\partial x}\left(x + \frac{F(x)}{f(x)}\right) \geqslant 0 \qquad \text{and} \qquad \frac{\partial}{\partial x}\left(x - \frac{1 + F(x)}{f(x)}\right) \geqslant 0. \tag{4}$$

Define the function B_0 by

$$B_0(x) = \arg\max_b v(b, x).$$

In the absence of an outside option, the proposer would bid $B_0(x)$. The following lemma characterizes the properties of B_0.

[14] The proposer can obtain arbitrarily close to these amounts and still give the chooser a strict incentive to choose as the proposer wishes. It is this indifference that does not extend to the asymmetric case.

[15] See McAfee and McMillan [5] for an interpretation of these conditions.

LEMMA 7. *$B_0(x)$ is the unique solution for b to $v_b(b, x)=0$. B_0 is non-decreasing. If $x<x_{\mathrm{m}}$, then $x<B_0(x)<x_{\mathrm{m}}$, and if $x>x_{\mathrm{m}}$, $x>B_0(x)>x_{\mathrm{m}}$. $B_0(x_{\mathrm{m}})=x_{\mathrm{m}}$.*

The equilibrium is efficient only if $B_0(x)=x$, which occurs only at the median value. If the proposer's value is less than the median of the chooser's value distribution, the proposer will obtain the good in circumstances when it would be efficient for the chooser to have the good. If the proposer's value is greater than the chooser's median value, the chooser will obtain the item in circumstances when it would be efficient for the proposer to have the item.

Now define x_r by $v(\max\{r, B_0(x_r)\}, x_r)=u(\frac{1}{2}r)$. The proposer will propose the outside option σ whenever his value $x_{\mathrm{p}}<x_r$. However, if $B_0(x)<r$, the proposer cannot offer his optimal bid and still receive the item. Thus, in this case, the proposer compares his utility of receiving the item, which requires bidding at least r, to the utility of bidding σ and obtaining $u(\frac{1}{2}r)$. This is summarized by the following theorem.

THEOREM 8. *The proposer offers σ if $x<x_r$. If $x>x_r$, the proposer offers $\max\{r, B_0(x)\}$. If $F(r)\geqslant\frac{1}{2}$, then $x_r=r$, and the outside option is exercised only when it is efficient to do so. If $F(r)<\frac{1}{2}$, then $x_r<r$, and the proposer keeps the item in circumstances when it would be efficient to exercise the outside option. For $x_r<r$, there is a discontinuity in bidding at x_r, as $B_0(x_r)>r>x_r$.*

Theorem 8 embodies an asymmetry which reflects the difference between an outside option value r below the median x_{m} and an outside option value r exceeding the median. If $r<x_{\mathrm{m}}$, the proposer would like to propose a payment in excess of his value. Thus, when he decides to not offer the outside option, he bids strictly in excess of his value, which exceeds the outside option value, and this creates a discontinuity at the switchover point x_r. If $r>x_{\mathrm{m}}$, if the proposer has a value near r, he would bid strictly less than his value, and this will be less than r. Thus, this bidder is forced to choose between bidding r to obtain the item and the outside option, because the preferred bid is not available. In this case, the proposer is better off with the outside option, rather than a bid of r, whenever the proposer's value is less than r.

In analyzing the CCM, so far only the distribution of the chooser's value and the utility function of the proposer has been used. To compare being the proposer to being the chooser, we impose symmetry and suppose that both agents are risk neutral and have the same distribution of independently distributed values.

THEOREM 9. *Suppose $r=0$ and the two agents are risk neutral and have* *identically and independently distributed valuations. Then the chooser has an interim utility strictly greater than that of the proposer, for every valuation.*

The CCM performs differently under asymmetric information than under full information in two significant respects. Allocative efficiency is lost. The strict preference to be the proposer under full information is reversed to a strict preference to be the chooser under asymmetric information.

Remark 5. It does not appear possible to rank the interim expected utility of the CCM with the utility produced by the WBA or the LBA generally. However, for the risk neutral case with a uniform distribution of values, the chooser in the CCM does better than an agent in the WBA, who does the same as an agent in the LBA, who does better than the proposer in the CCM.

3. Allocations without Transfers: The Alternating Selection Mechanism

Very casual empiricism suggests that the alternating selection mechanism, where one agent chooses one item, then the second chooses one of the remainder, then the first chooses again, and so on,[16] is frequently used to allocate a large number of items of roughly similar value. The ASM has two main advantages. First, it does not involve transfers and pricing at all. This can be important when the laws, such as California's community property divorce law, require equal division of the property, as judged by value. Since bids are a natural notion of value, courts may not allow divisions that would arise under bidding or order additional transfers to equate the monetary value of the division.[17] The ASM sidesteps such considerations. Second, and more importantly, the ASM can be operated easily and quickly, avoiding the complex calculations required for optimal bidding by simplifying the strategy space. If the ASM is reasonably efficient, then its ease of use would explain its popularity.

Fix the number of items to be allocated at n, with items identified with integers, and let $X_1, ..., X_n$ be the values that agent 1 holds for the items and $Y_1, ..., Y_n$ be the values held by agent 2. I shall use lowercase x_i and y_i for the realizations of the random variables X_i and Y_i, respectively, and X and Y for $(X_1, ..., X_n)$ and $(Y_1, ..., Y_n)$, respectively. Agent 1 is informed

[16] This mechanism is used by children to allocate players to baseball teams quite frequently as well.

[17] For example, if the parties use a loser's bid auction, the court may decide that the value is at least the winner's bid and ex post order the winner to pay the loser half the winning bid. California courts have repeatedly demonstrated a willingness to alter the division of property *even when both parties agreed to the division.*

privately about $x_1, \ldots, x_n$ and agent 2 is privately informed about $y_1, \ldots, y_n$ at the start of the game. The distributions of the players' values and the rules of the game are common knowledge. The ASM is an n-stage game. At stage k, there is a set $S_k \subseteq \{1, \ldots, n\}$ of available items. In odd numbered periods k, agent 1 selects an item from S_k, and in even numbered periods k, agent 2 selects an item from S_k, and $S_{k+1} = S_k \setminus \{l\}$, where l is the item chosen in period k. If σ_i is the set of items that agent i selects, then the payoff to 1 is $\sum_{k \in \sigma_1} x_k$ and agent 2's payoff is $\sum_{k \in \sigma_2} y_k$.

It is of course possible that an agent will not select his most preferred item. For example, if $n=3$, $x_1 > x_2 > x_3$, and $y_2 > y_3 > y_1$, agent 1 does better to select item 2 than item 1 in the first stage, provided he knows agent 2's preferences. Such an effect depends critically on the agents knowing each others' preferences, and in the first result of this section, I give a set of circumstances when it is an equilibrium for both agents to select their most preferred item in all instances. Call this myopic strategy, of picking one's most preferred item, the *rank selection strategy*, or RSS.

3.1. *The RSS is an Equilibrium*

Assume that (X, Y) has a density, so that the probability that $x_i = x_j$ or $y_i = y_j$ is zero, for $i \neq j$. Then I can write the agent 1's prior probability that agent 2 prefers item $k \in S$ when the set of available items is S as $P(Y_k \geqslant \max_{i \in S} Y_i \mid X = x)$. Generally, I will suppress the dependence of this probability on the realization of $X = (X_1, \ldots, X_n)$ and leave this dependence implicit.

ASSUMPTION 1 (Rank Independence). *Conditioned on any realization of X,*

$$\text{for } S \subseteq T, k \in S, j \in T \setminus S, \; P(Y_k \geqslant \max_{i \in S} Y_i \mid Y_j \geqslant \max_{i \in T} Y_i) = P(Y_k \geqslant \max_{i \in S} Y_i). \tag{5}$$

This property is also assumed to hold with X and Y interchanged.

Assumption 1 allows a dramatic simplification in the description of agent 1's perception of agent 2's preferences, because agent 1 does not revise his perception of agent 2's ranking of the remaining items when he sees agent 2's selections, given that agent 2 uses the RSS. This is formalized in the following lemma.

LEMMA 10. *There are probabilities $q_1, \ldots, q_n$, which depend on x, so that*

$$P(Y_k \geqslant \max_{i \in S} Y_i) = \frac{q_k}{\sum_{i \in S} q_i}. \tag{6}$$

Assumption 1 serves two distinct roles in simplifying this problem. First, it preserves order. If agent 1 thinks agent 2 is more likely to prefer item i over j when k is available, then agent 1 will continue to believe this after k is chosen by agent 2, a strong form of independence of irrelevant alternatives. Thus, Assumption 1 eliminates the dependence of profits on the history of choices, by eliminating learning, provided that the agents use the RSS. Therefore, to show that the RSS is a best response to the RSS, I may hold the relative likelihood of the other agent's ranking constant. Moreover, agent 1 has a well-defined expected profit function that depends only on the realization of X and the set of available items, but not on the history of agent 2's choices, because agent 1 holds the same posterior distribution of the ranking of agent 2's remaining choices after seeing agent 2's earlier selections, provided agent 2 uses the RSS. In particular, it will never be in agent 2's interest to select an item that he does not value highly solely to alter agent 1's posterior distribution about agent 2's ranking of the items, as might occur if Assumption 1 failed to hold. This logic also applies to agent 2's posterior probabilities about agent 1, of course.

Assumption 1 is unlikely to be satisfied in many circumstances, for generally an agent's choices reveal something about the agent's preferences for types of items. Consider the dissolution of a medical practice. If one doctor observes the other doctor choosing only heart problems, the first doctor might reasonably conclude that the second doctor is attempting to specialize, and hence the first doctor should believe it safe to delay picking the patients with broken bones. Similarly, if a collection of compact discs is being divided and one person picks only jazz CDs, the other might reasonably deduce a heretofore unsuspected preference for jazz. In some circumstances, however, Assumption 1 is not unreasonable. For example, if a brokerage or insurance partnership is allocating the clients among the partners, the partners might reasonably believe that the values of different clients are independently distributed. The values held by different partners for the same client will not be independently distributed, because these values reflect the circumstances of the client. However, the type of client will be less important in such circumstances. Indeed, the only consideration may be how much each partner expects to be able to sell to that client. This depends on the partner and on the client, but not on the other clients.

ASSUMPTION 2 (Agreement on Ranking).

$$\textit{If } x_i \leqslant x_j, \textit{ then } q_i \leqslant q_j. \tag{7}$$

The equivalent restriction is assumed to hold for agent 2's priors as well.

In order for the RSS to be a sensible strategy, an agent who forgoes his preferred item should think that the other agent is likely to choose that preferred item. It is this consideration that induces agents to select their most preferred item, as is embodied in Assumption 2. In particular, if agent 1 thinks that it is very unlikely that agent 2 likes agent 1's most preferred good, agent 1 may delay choosing his most preferred good in order to obtain less preferred goods. Even if $n = 3$, agent 1 will select his second most preferred good over his most preferred good, provided the probability that agent 2 prefers the second most preferred good is sufficiently high.

Without loss of generality, I can name the existing alternatives according to agent 1's preferences, so that $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_m$ when there are m available alternatives, and let $S = \{1, ..., m\}$. For any set $S \subseteq \{1, ..., n\}$, there are continuation values for agent 1, denoted $\pi_1(S; X)$, if it is agent 1's turn to select,[18] and $\pi_2(S; X)$, if it is agent 2's turn to select. Note that $\pi_2(S; X)$ is agent 1's payoff when he is not selecting, and not agent 2's payoff. These continuation values depend on the q_i's, of course. I shall suppress X in $\pi_i(S; X)$ because it will remain constant, once realized at the beginning of the game. Provided both agents use the RSS, the continuation values obey the recursive formulas

$$\pi_1(S) = x_m + \pi_2(S \backslash \{m\}), \tag{8}$$

and

$$\pi_2(S) = \sum_{i \in S} \pi_1(S \backslash \{i\}) \frac{q_i}{\sum_{j \in s} q_j}, \tag{9}$$

where q_i is the value guaranteed by Lemma 6. Equation (8) arises because agent 1 chooses his most highly valued item, and then it is agent 2's turn. Equation (9) arises because agent 2 will select his most highly valued item i with the probability given in Lemma 6, and then it is agent 1's turn to select from the set $S \backslash \{i\}$.

There is a peculiar feature to this problem that was unexpected and seems to work as follows. If agent 1 chooses m, his most preferred item, from S, it increases the likelihood that he reaches the continuation value $\pi_2(S \backslash \{m, m-1, j\})$ relative to choosing any other item k, other than j or m, because $q_j / \sum_{i \in S \backslash \{m\}} q_i \geqslant q_j / \sum_{i \in S \backslash \{k\}} q_i$. But this suggests that showing the RSS is an equilibrium requires one to prove the intuitively plausible formula $\pi_2(S \backslash \{m\}) \leqslant \pi_2(S \backslash \{j\})$, for $j \leqslant m$. However, this is not true at $n = 3$ without further hypotheses, while the RSS is an equilibrium for $n = 3$ under Assumption 1 and 2. $\pi_2(S \backslash \{m\}) \leqslant \pi_2(S \backslash \{j\})$ is stronger than

[18] It is agent 1's turn to select in odd numbered periods, numbering from the beginning. However, such numbering is not very useful.

necessary to show the RSS is an equilibrium. The useful condition is $\pi_2(S\setminus\{m\})-x_m\leqslant\pi_2(S\setminus\{j\})-x_j$, but even this requires a further hypothesis. Consequently, I introduce one further assumption:

ASSUMPTION 3. *Number the m available items according to agent* 1*'s preferences, so that* $x_1\leqslant\cdots\leqslant x_m$. *Let* q_i *be as in Lemma* 10, $p_i=q_i/\sum_{j=1}^m q_j$, *and* $k<m$. *Then*

$$\sum_{j\neq m}\frac{x_j p_j}{1-p_m}-\sum_{j\neq k}\frac{x_j p_j}{1-p_k}+2\left[\frac{p_m(x_m-x_{m-1})}{1-p_k}+\frac{p_k(x_{m-1}-x_k)}{1-p_m}\right]\geqslant 0. \quad (10)$$

This is also presumed to hold with the X's and Y's interchanged.

About the only good thing that can be said for Assumption 3 is that it tends to hold strictly when Y is distributed independently of X. It should be noted that Assumptions 1, 2, and 3 bound the correlation between Y_i and X_i. Sufficient conditions for (10) are given in the following lemma:

LEMMA 11. *Suppose* $0\leqslant x_1\leqslant\cdots\leqslant x_n$ *and* $0\leqslant q_1\leqslant\cdots\leqslant q_n$, $\sum_{i=1}^n q_i=1$, *and* $(\forall k\leqslant l)$ $(q_k(1-q_k))/(q_l(1-q_l))\leqslant 2-(x_k/x_l)$. *Then* (10) *holds for all* $k\leqslant m\leqslant n$.

Lemma 11 provides a sufficient condition on the mapping of x to q which guarantees that Assumption 3 holds. Given that $x_k\leqslant x_l$, Assumption 2 puts a lower bound on q_l, and Assumption 3 puts an upper bound on q_l.[19]

THEOREM 12. *Given Assumptions* 1, 2, *and* 3, *the RSS forms a sequential equilibrium. Moreover, if* $x_m=\max_{j\in S}x_j$ *and* $k\in S$, *then*

$$x_m-x_k\geqslant\pi_2(S\setminus\{m\})-\pi_2(S\setminus\{k\})\geqslant x_k-x_m. \quad (11)$$

Assumptions 1, 2, and 3 are satisfied when the random variables $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ are independently distributed, and I now turn to this case.

3.2. *Identically and Independently Distributed Valuations*

Let F be the distribution function of independently distributed valuations, with density f. Let $\alpha_{k,n}$ be the probability that agent 1, who chooses first of the n items, obtains his kth preferred good when there are n goods available. Thus, if agent 1 ranks the goods so that i_j is preferred to i_{j+1}, then $\alpha_{k,n}$ is the probability that agent 1 will obtain good i_k. Similarly, $\beta_{k,n}$ is used to denote the probability that agent 2 obtains agent 2's kth most

[19] Although the bound given in Lemma 11 is not strict, it is easily verified that Assumption 3 puts an upper bound on q_1.

preferred good. These probabilities depend only on the rank of the good and not on the strength of preferences, because both players use the RSS.

LEMMA 13. *For* $n \geqslant 1$, $\alpha_{1,n} = 1$ *and* $\beta_{1,n} = (n-1)/n$. $\alpha_{2,2} = 0$ *and* $\beta_{2,2} = \frac{1}{2}$. *For* $k \geqslant 2$ *and* $n \geqslant 3$,

$$\alpha_{k,n} = \frac{k-2}{n-1} \alpha_{k-2,n-2} + \frac{n-k}{n-1} \alpha_{k-1,n-2}$$

and

$$\beta_{k,n} = \frac{k-1}{n} \alpha_{k-1,n-1} + \frac{n-k}{n} \alpha_{k,n-1}.$$

In addition, for n *even,* $\alpha_{k,n} = (n-k)/(n-1)$ *and* $\beta_{k,n} \geqslant (n-k)/n$. *For* n *odd,* $\alpha_{k,n} \geqslant (n-k)/(n-1)$ *and* $\beta_{k,n} = (n-k)/n$.

There is an interesting difference between the case when n is even and when n is odd, embodied in Lemma 13. When n is even, there is a simple expression for $\alpha_{k,n}$, but not when n is odd. This difference seems to have something to do with choosing last and hence getting whatever is left for one from the previous selections.

THEOREM 14. *The total expected surplus from the full information efficient allocation is* $T = n \int_0^{x_H} 1 - F(x)^2 \, dx$. *If* n *is even, the first agent's expected utility in the* ASM *is* $\frac{1}{2}T$, *and the expected utility of the second agent exceeds* $(n-1)\,T/(2n)$. *The expected loss in the* ASM, *as compared to the efficient allocation, does not exceed* $\frac{1}{2}x_H$, *independent of* n.

In the case of independence, the ASM is remarkably efficient. For n even, the agent who chooses first gets the same utility level that he would get under an even split of the utility generated in a full information efficient solution. The other agent gets within $1/(2n)$ of this level and bears the entire cost of the inefficiency. This loss, of course, does not exceed $\frac{1}{2}$ the maximal value of one item, which is relatively trivial when there are many items. The case of odd n is similar.

For uniform $[0, 1]$ distribution of values and $n = 12$, the expected full information surplus is 8, and the ASM produces an expected surplus of 7.74, or 96.8%. To illustrate the misbehavior of β for even n, I provide the values for $n = 12$. Recall that $\beta_{k,12}$ is the probability that the agent who chooses second gets his kth favorite item of 12:

k	1	2	3	4	5	6	7	8	9	10	11	12
$\beta_{k,12}$	$\frac{11}{12}$	$\frac{10}{12}$	$\frac{9}{12}$	$\frac{8}{12}$	$\frac{7}{12}$	$\frac{6}{12}$	$\frac{161}{384}$	$\frac{133}{384}$	$\frac{147}{512}$	$\frac{63}{256}$	$\frac{231}{1024}$	$\frac{231}{1024}$

This illustrates that the probability of acquiring the nth and the $(n-1)$st favored objects must coincide for the agent who selects last.

4. Conclusion

Given the restrictions of the model, the winner's bid auction appears to be the best way to allocate a single item, because it reaches an ex post efficient allocation, even in the presence of an outside option. The assumptions are, unfortunately, extremely restrictive. In particular, the independence assumption is dificult to defend in most applications. The winner's bid auction also has the appealing property that it levels out utility relative to the loser's bid auction, which leads me to conjecture that the winner's bid auction brings a higher expected utility than the loser's bid auction, although I am unable to prove this.

The cake-cutting mechanism has a disappointing performance in this environment, as it fails to reach ex post efficiency. This result casts a shadow on the entire literature on cake-cutting type mechanisms. A reasonable conjecture is that the generalizations of the CCM will have a performance poorer than that of other more symmetric mechanisms when asymmetric information is introduced, which provides an exciting research agenda. In general, little is known about the generalization of the *fairness* literature to the asymmetric information environment, and the conclusions of this paper suggest that this generalization could change our understanding of fairness. However, this remains an open question.

It seems interesting to know what can be accomplished without transfers, but only when this simplifies the mechanism, because the object of eliminating transfers is to produce simple mechanisms. In a very special environment, the ASM possesses both simple strategies and a remarkably efficient outcome. One objection to using simple mechanisms such as the WBA and the LBA is that, while the mechanism is simple, the equilibrium strategies of the bidders are not simple, and restrictive assumptions on the environment are required to obtain closed form strategies. Under other restrictive assumptions, the ASM produces simple myopic strategies, which is obviously a good thing. It would be nice to have a full characterization of the equilibrium strategies in a general affiliated environment.

Finally, it is somewhat unsatisfactory to assume that the mechanisms that are used will be simple ones, even if one can make a strong case for this. It would be nice to deduce the use of simple mechanisms in a manner analogous to the optimality of linear contracts in Holmstrom and Milgrom [3], Laffont and Tirole [4], or McAfee and McMillan [6]. Finding the restriction that leads to the optimality of simple mechanisms seems to me to be the most important problem facing mechanism design.

APPENDIX: PROOFS

Because we shall be verifying that several bidding functions compose equilibria, it is useful to begin with the following lemma. It is simple to prove, and a proof is found in McAfee [7].

LEMMA 0. *Suppose an agent of type x who reports a type y obtains* $v(y, x)$, *and* v *is* C^2. *This agent will choose to be honest for every type x if*

$$(\forall x) \qquad \frac{\partial v}{\partial y}(x, x) = 0 \tag{A1}$$

and

$$(\forall x)(\forall y) \qquad \frac{\partial^2 v}{\partial y\, \partial x}(x, y) \geqslant 0. \tag{A2}$$

Moreover, (A1) *is necessary, and*

$$(\forall x) \qquad \frac{\partial^2 v}{\partial x\, \partial y}(x, x) \geqslant 0 \tag{A3}$$

is also necessary.

The use of Lemma 0 is as follows. Suppose a bidder with value x who bids b anticipates profits $\pi(x, b)$. Then B is an equilibrium bidding function if $v(y, x) = \pi(x, B(y))$ is maximized over y at $y = x$; that is, the bidder chooses to be honest in the direct mechanism version of the bidding game.

Proof of Lemma 1. Consider a bidder with value x who submits a bid of $B_w(y)$. This bidder receives x-$\frac{1}{2}B_w(y)$ with probability $F(y)$ and otherwise receives $\frac{1}{2}B_w(s)$, where $s > y$ is the other bidder's value, with density $f(s)$. The bidder's expected utility is

$$v(y, x) = \begin{cases} \lambda^{-1}\left[1 - e^{-(1/2)\lambda r} F(r) - \int_r^{x_H} e^{-(1/2)\lambda B_w(s)} f(s)\, ds\right] & \text{if } y \leqslant r \\ \lambda^{-1}\left[1 - e^{-\lambda[x - (1/2)B_w(y)]} F(y) - \int_y^{x_H} e^{-(1/2)\lambda B_w(s)} f(s)\, ds\right] & \text{if } y \geqslant r. \end{cases}$$

For $x \geqslant r$, by (2),

$$e^{\lambda B_w(x)} = F(x)^{-2}\left[e^{\lambda r} F(r)^2 + \int_r^x e^{\lambda s} 2F(s) f(s)\, ds\right], \tag{A5}$$

$$e^{\lambda B_w(x)} \lambda B_w'(x) = \frac{2f(x)}{F(x)}\left[e^{\lambda x} - e^{\lambda B_w(x)}\right], \tag{A6}$$

and

$$B'_{\rm w}(x) = \frac{2f(x)}{\lambda F(x)} \, [e^{\lambda(x - B_{\rm w}(x))} - 1]. \tag{A7}$$

For $x > r$,

$$\frac{\partial v\,(y, x)}{\partial y} = \lambda^{-1} \Bigg[-e^{-\lambda(x - (1/2)\,B_{\rm w}(y))} f(y) - e^{-\lambda(x - (1/2) B_{\rm w}(y))} \frac{1}{2} \lambda B'_{\rm w}(y)\, F(y) + e^{-(1/2)\,\lambda B_{\rm w}(y)} f(y) \Bigg]$$

$$= \lambda^{-1} f(y)\, e^{-(1/2)\lambda B_{\rm w}(y)} [1 - e^{\lambda(y-x)}]. \tag{A8}$$

(A1) and (A2) follow immediately from (A8), so by Lemma 0, $B_{\rm w}(x)$ is an optimal bid for a buyer with value $x > r$. It is easily verified that $B_{\rm w}(r) = r$, and thus that a bidder with valuation less than r will also choose to bid σ, which means that $B_{\rm w}$ is a symmetric Nash equilibrium bidding function. Note, however, that for $x < r$, any bid not exceeding r will do.

$B_{\rm w}$ is increasing, provided $B_{\rm w}(x) < x$, from (A6). This is guaranteed by

$$e^{\lambda B_{\rm w}(x)} = F(x)^{-2} \left[e^{\lambda r} F(r)^2 + \int_r^x e^{\lambda s} 2F(s) f(s)\, ds \right]$$

$$< F(x)^{-2} \left[e^{\lambda x} F(r)^2 + \int_r^x e^{\lambda x} 2F(s) f(s)\, ds \right] = e^{\lambda x}. \quad \blacksquare$$

Proof of Lemma 2. First, note that

$$\frac{\partial B_{\rm w}(x)}{\partial \lambda} = \lambda^{-1} \left[-B_{\rm w}(x) + \frac{r e^{\lambda r} F(r)^2 + \int_r^x e^{\lambda s} s 2F(s) f(s)\, ds}{e^{\lambda r} F(r)^2 + \int_r^x e^{\lambda s} 2F(s) f(s)\, ds} \right] \geqslant 0. \tag{A9}$$

To establish the inequality, first note that $Y \log Y$ is convex in Y for $Y \geqslant e^{-1}$, so that, for any random variable with support contained in $[e^{-1}, \infty)$, $E(Y \log Y) \geqslant (EY) \log(EY)$, where E is expectation. Now define a random variable Z which takes on the value r with probability $[F(r)/F(x)]^2$ and takes on the the value $z \in (r, x]$ with density $2F(z) f(z)/F(x)^2$. Note that $B_{\rm w}(x) = \lambda^{-1} \log(Ee^{\lambda z})$. Now let Y be $e^{\lambda z}$, which guarantees that the support of Y is contained in $[e^{-1}, \infty)$. Thus $Ee^{\lambda z} \lambda Z \geqslant (Ee^{\lambda z}) \log(Ee^{\lambda z})$, which reduces to (A9), given the definition of Z.

The limiting value of $B_{\rm w}(x)$ is a straightforward application of L'Hopital's rule on (2).

Finally, the expected utility of a bidder with valuation x is

$$\begin{aligned}
v(x, x) &\geqslant v(B_w^{-1}(x), x) \\
&= \lambda^{-1}\left[1 - e^{-\lambda[x-(1/2)x]}F(B_w^{-1}(x)) - \int_{B_w^{-1}(x)}^{x_H} e^{-(1/2)\lambda B_w(y)} f(y)\, dy\right] \\
&\geqslant \lambda^{-1}\left[1 - e^{-(1/2)\lambda x}F(B_w^{-1}(x)) - \int_{B_w^{-1}(x)}^{x_H} e^{-(1/2)\lambda x} f(s)\, ds\right] \\
&= \lambda^{-1}[1 - e^{-(1/2)\lambda x}] = u\left(\tfrac{1}{2}x\right).
\end{aligned}$$

Moreover, strict inequality holds for all x, either at the first inequality, if $x > r$, or at the second, if $x < x_H$. ∎

Proof of Lemma 3. Let $x_r = \inf\{x : B_1(x) > r\}$. The expected utility of a bidder with value x who bids $B_1(x)$ is

$$v(y, x) = \begin{cases} \lambda^{-1}\left[1 - \int_0^y e^{-\lambda(x-(1/2)B_1(s))} f(s)\, ds - (1 - F(y))\, e^{-(1/2)\lambda B_1(y)}\right] & \text{if } y > x_r \\ \lambda^{-1}[1 - e^{(1/2)\lambda r}] & \text{if } y \leqslant x_r. \end{cases} \tag{A10}$$

Note that a bidder will never bid less than r, since a bid of r produces a payoff of $\frac{1}{2}r$, while a bid $b < r$ produces a payoff of $\frac{1}{2}r$ with probability $F(x_r)$ and of $\frac{1}{2}b$ with probability $1 - F(x_r)$.

So consider reports $y > x_r$. Then,

$$\begin{aligned}
\frac{\partial v\,(y, x)}{\partial y} &= f(y)\, \lambda^{-1}[e^{-(1/2)\lambda B_1(y)} - e^{-\lambda(x-(1/2)B_1(y))}] \\
&\quad + (1 - F(y))\, e^{-(1/2)\lambda B_1(y)}\, \tfrac{1}{2} B_1'(y) \\
&= f(y)\, e^{(1/2)\lambda B_1(y)} \lambda^{-1} [e^{-\lambda B_1(y)} - e^{-\lambda x}] \\
&\quad + (1 - F(y))\, e^{-(1/2)\lambda B_1(y)}\, \tfrac{1}{2} B_1'(y).
\end{aligned} \tag{A11}$$

Clearly, $\partial^2 v / \partial x\, \partial y > 0$. Noting that

$$e^{-\lambda B_1(x)} = (1 - F(x))^{-2} \int_x^{x_H} e^{-\lambda s} 2(1 - F(s)) f(s)\, ds, \tag{A12}$$

we have

$$-e^{-\lambda B_1(x)}\lambda B_1'(x)=\frac{2f(x)}{(1-F(x))}\,[e^{-\lambda B_1(x)}-e^{-\lambda x}]. \tag{A13}$$

Substitution of (A13) into (A11), setting $y=x$, yields $(\partial v/\partial y)(x, x)=0$. Thus, by Lemma 0, B_1 is an equilibrium bidding function. By (A12), we see that $B_1(x)>x$ for $x<x_{\mathrm{H}}$, and in particular for $x=r$. By (A13), we have that $B_1'(x)>0$. ∎

Proof of Lemma 4. This proof is virtually identical to the proof of Lemma 2 and is omitted.

Proof of Theorem 5.

$$\begin{aligned}
\pi_{\mathrm{w}}(x)|_{\lambda=0}&=\left(x-\frac{1}{2}B_{\mathrm{w}}(x)\right)F(x)+\frac{1}{2}\int_x^{x_{\mathrm{H}}}B_{\mathrm{w}}(y)f(y)\,dy\bigg|_{\lambda=0}\\
&=\frac{1}{2}F(x)\left(x+F(x)^{-2}\int_0^x F(y)^2\,dy\right)\\
&\quad+\frac{1}{2}\int_x^{x_{\mathrm{H}}}\left(y-F(y)^{-2}\int_0^y F(s)^2\,ds\right)f(y)\,dy\\
&=\frac{1}{2}\left[xF(x)-F(x)^{-1}\int_0^x F(y)^2\,dy+yF(y)\bigg|_x^{x_{\mathrm{H}}}\right.\\
&\quad\left.-\int_x^{x_{\mathrm{H}}}F(y)\,dy-\left(\left(\int_0^y F(s)^2\,ds(-F(y)^{-1})\right)\bigg|_x^{x_{\mathrm{H}}}+\int_x^{x_{\mathrm{H}}}F(y)\,dy\right)\right]\\
&=\frac{1}{2}\left[x_{\mathrm{H}}-2\int_x^{x_{\mathrm{H}}}F(y)\,dy+\int_0^{x_{\mathrm{H}}}F(y)^2\,dy\right]
\end{aligned}$$

$$\begin{aligned}
\pi_1(x)|_{\lambda=0}&=\int_0^x\left[x-\frac{1}{2}B_1(y)\right]f(y)\,dy+\frac{1}{2}B_1(x)(1-F(x))\\
&=xF(x)-\frac{1}{2}\int_0^x\left(y+(1-F(y))^{-2}\int_y^{x_{\mathrm{H}}}(1-F(s))^2\,ds\right)f(y)\,dy\\
&\quad+\frac{1}{2}(1-F(x))\left[x+(1-F(x))^{-2}\int_x^{x_{\mathrm{H}}}(1-F(y))^2\,dy\right]\\
&=xF(x)-\frac{1}{2}\left[xF(x)-\int_0^x F(y)\,dy+(1-F(y))^{-1}\right.\\
&\quad\left.\times\int_y^{x_{\mathrm{H}}}(1-F(s))^2\,ds\bigg|_0^x+\int_0^x 1-F(y)\,dy\right]\\
&\quad+\frac{1}{2}\left[x(1-F(x))+(1-F(x))^{-1}\int_x^{x_{\mathrm{H}}}(1-F(y))^2\,dy\right]
\end{aligned}$$

$$=\frac{1}{2}\left[xF(x)+\int_0^x F(y)\,dy+\int_0^{x_{\mathrm{H}}}(1-F(s))^2\,ds\right.$$
$$\left.-\int_0^x 1-F(y)\,dy+x(1-F(x))\right]$$
$$=\frac{1}{2}\left[2\int_0^x F(y)\,dy+\int_0^{x_{\mathrm{H}}}(1-F(s))^2\,ds\right]$$
$$=\frac{1}{2}\left[x_{\mathrm{H}}+\int_0^{x_{\mathrm{H}}}F(s)^2\,ds-2\int_x^{x_{\mathrm{H}}}F(s)\,ds\right]$$
$$=\pi_{\mathrm{w}}(x)|_{\lambda=0}. \quad \blacksquare$$

Proof of Lemma 7. Note that

$$v_b=-\tfrac{1}{2}u'(x-\tfrac{1}{2}b)\,F(b)+u(x-\tfrac{1}{2}b)\,f(b)$$
$$+\tfrac{1}{2}u'(\tfrac{1}{2}b)(1-F(b))-u(\tfrac{1}{2}b)\,f(b)$$
$$=f(b)\left[u(x-\tfrac{1}{2}b)-\tfrac{1}{2}u'(x-\tfrac{1}{2}b)\frac{F(b)}{f(b)}\right.$$
$$\left.-\left(u(\tfrac{1}{2}b)-\tfrac{1}{2}u'(\tfrac{1}{2}b)\frac{1-F(b)}{f(b)}\right)\right].$$

Thus $v_b(b,x)=0$ implies

$$v_{bb}(b,x)=f(b)\left[-\tfrac{1}{2}u'(x-\tfrac{1}{2}b)\frac{\partial}{\partial b}\left(b+\frac{F(b)}{f(b)}\right)\right.$$
$$+\tfrac{1}{4}u''(x-\tfrac{1}{2}b)\frac{F(b)}{f(b)}-\tfrac{1}{2}u'(\tfrac{1}{2}b)\frac{\partial}{\partial b}\left(b-\frac{1-F(b)}{f(b)}\right)$$
$$\left.+\tfrac{1}{4}u''(\tfrac{1}{2}b)\frac{1-F(b)}{f(b)}\right]<0.$$

Thus there is at most one solution to $v_b(b,x)=0$. Moreover,

$$v_b(0,x)=(u(x)-u(0))\,f(0)+\tfrac{1}{2}u'(0)>0$$

and

$$v_b(x_{\mathrm{H}},x)=-\tfrac{1}{2}u'(x-\tfrac{1}{2}x_{\mathrm{H}})-f(x_{\mathrm{H}})[u(\tfrac{1}{2}x_{\mathrm{H}})-u(x-\tfrac{1}{2}x_{\mathrm{H}})]<0.$$

since $\tfrac{1}{2}x_{\mathrm{H}}>x-\tfrac{1}{2}x_{\mathrm{H}}$, as $x<x_{\mathrm{H}}$.

It is useful to note that $b > B_0(x)$ implies $v_b(b, x) < 0$ and $b < B_0(x)$ implies $v_b(b, x) > 0$, which will be useful in the proof of Theorem 8. $v_{xb} = -\frac{1}{2}u''(x - \frac{1}{2}b)\,F(b) + u'(x - \frac{1}{2}b)\,f(b) > 0$, so B_0 is increasing.

Now suppose $B_0(x) > x$. Then $u(x - \frac{1}{2}B_0) < u(\frac{1}{2}B_0(x))$ and $u'(x - \frac{1}{2}B_0) > u'(\frac{1}{2}B_0(x))$. Thus

$$\begin{aligned} 0 &= \tfrac{1}{2}[u'(\tfrac{1}{2}B_0(x))(1 - F(B_0(x))) - u'(x - \tfrac{1}{2}B_0(x))\,F(B_0(x))] \\ &\quad + f(B_0(x))\{u(x - \tfrac{1}{2}B_0(x)) - u(\tfrac{1}{2}B_0(x))\}] \\ &< \tfrac{1}{2}[u'(\tfrac{1}{2}B_0(x))(1 - F(B_0(x))) - u'(x - \tfrac{1}{2}B_0(x))\,F(B_0(x))] \\ &< \tfrac{1}{2}u'(x - \tfrac{1}{2}B_0(x))[1 - 2F(B_0(x))]. \end{aligned}$$

Thus $F(B_0(x)) < \frac{1}{2}$; that is, $B_0(x) > x$ implies $B_0(x) < x_{\rm m}$. Similarly, by reversing the inequalities above, we obtain $B_0(x) < x$ implies $B_0(x) > x_{\rm m}$. Finally, $v_b(x_{\rm m}, x_{\rm m}) = 0$, so $B_0(x_{\rm m}) = x_{\rm m}$. ∎

Proof of Theorem 8.

Case 1. $r > x_{\rm m}$. For $x < r$, $B_0(x) < r$, and thus, for all $b \geq r$, using the fact stated in the proof of Lemma 7, $v(b, x) \leq v(r, x) = u(x - \frac{1}{2}r)\,F(r) + u(\frac{1}{2}r)(1 - F(r)) < u(\frac{1}{2}r)$. For $x > r$, $v(r, x) = u(x - \frac{1}{2}r)\,F(r) + u(\frac{1}{2}r)(1 - F(r)) > u(\frac{1}{2}r)$. Thus the proposer proposes the bid σ if $x < r$ and a bid of $\max\{r, B_0(x)\}$ if $x > r$.

Case 2. $r < x_m$. Define x_r by $v(B_0(x_r), x_r) = u(\frac{1}{2}r)$. Note that $v(B_0(x), x)$ is increasing in x, since $v_x(b, x) > 0$. As $v(B_0(r), r) > v(r, r) = u(\frac{1}{2}r)$, $x_r < r$. Now let x_a satisfy $B_0(x_a) = r$, should this equation have a solution, and otherwise $x_a = 0$. $x_a < r$ since $r < x_m$. Then $v(B_0(x_a), x_a) = v(r, x_a) < v(r, r) = u(\frac{1}{2}r)$. Thus $x_r \in (x_a, r)$. For $x < x_r$, $v(B_0(x), x) < u(\frac{1}{2}r)$, and the proposer proposes σ. For $x > x_r$, $v(B_0(x), x) > u(\frac{1}{2}r)$, and the proposer proposes $b \geq r$. Since $x_r > x_a$, $B_0(x_r) > r$, and there is a jump discontinuity in the equilibrium bidding function at x_r. ∎

Proof of Theorem 9. Under risk neutrality, the proposer's expected utility is

$$\pi_{\rm p}(x) = (x - \tfrac{1}{2}B_0(x))\,F(B_0(x)) + \tfrac{1}{2}B_0(x)(1 - F(B_0(x)))$$

and thus

$$\pi_{\rm p}'(x) = F(B_0(x)).$$

The chooser's expected utility is

$$\pi_{\rm c}(x) = E\max\{x - \tfrac{1}{2}B_0(x_p), \tfrac{1}{2}B_0(x_p)\}$$

and thus

$$\pi_c'(x)=\begin{cases}0 & \text{if}\quad x\leqslant B_0(0)\\ F(B_0^{-1}(x)) & \text{if}\quad B_0(0)<x<B_0(x_H)\\ 1 & \text{if}\quad B_0(x_H)\leqslant 0.\end{cases}$$

For $x\in(B_0(0), B_0(1))$, $\pi_p'(x)>\pi_c'(x)$ if and only if $B_0(x)>B_0^{-1}(x)$ if and only if $x<x_m$. Thus, $\pi_c-\pi_p$ is minimized at x_m. $\pi_c(x_m)=E\max\{x_m-\frac{1}{2}B_0(x_p), \frac{1}{2}B_0(x_p)\}>\frac{1}{2}x_m=\pi_p(x_m)$, and thus $(\forall x)\,\pi_c(x)>\pi_p(x)$. ▮

Proof of Lemma 10. The proof is by induction on the cardinality of S. Given the realization of X, let $q_i=P(Y_i\geqslant\max_{j\neq i}Y_j)$, and suppose (6) holds for all S with cardinality exceeding m. Equation (6) is true when the cardinality of S is n by the definition of q_i. We show that (6) holds for S with m members. Let l be an item *not* in S. Then

$$\begin{aligned}P(Y_k\geqslant\max_{i\in S}Y_i)&=P(Y_k\geqslant\max_{i\in S\cup\{l\}}Y_i)\\&\quad+P(Y_k\geqslant\max_{i\in S}Y_i\mid Y_l\geqslant\max_{i\in S\cup\{l\}}Y_i)\,P(Y_l\geqslant\max_{i\in S\cup\{l\}}Y_i)\\&=\frac{q_k}{\sum_{i\in S\cup\{l\}}q_i}+P(Y_k\geqslant\max_{i\in S}Y_i)\frac{q_l}{\sum_{i\in S\cup\{l\}}q_i}.\end{aligned}$$

Solving for $P(Y_k\geqslant\max_{i\in S}Y_i)$ gives the result. ▮

Proof of Lemma 11. Let $p_j=q_j/\sum_{i\in S}q_i$. Note that

$$\frac{p_m}{1-p_k}=1-\sum_{j\neq k,m}\frac{p_j}{1-p_k}\geqslant 1-\sum_{j\neq k,m}\frac{p_j}{1-p_m}=\frac{p_k}{1-p_m}$$

and

$$\frac{p_m(1-p_m)}{p_k(1-p_k)}=\frac{q_m(\sum_{i\in S}q_i-q_m)}{q_k(\sum_{i\in S}q_i-q_k)}\leqslant\frac{q_m(1-q_m)}{q_k(1-q_k)}\leqslant 2-\frac{x_k}{x_m}.$$

The term in square brackets in (10) is nonincreasing in x_{m-1} and it is sufficient to show that (10) holds setting $x_{m-1}=x_m$ on the right-hand side. Multiplying by $(1-p_k)(1-p_m)$ gives

$$\begin{aligned}&(1-p_k)\sum_{j\neq m}x_j p_j-(1-p_m)\sum_{j\neq k}x_j p_j+2p_k(1-p_k)(x_m-x_k)\\&=(1-p_k)\left[x_k p_k+\sum_{j\neq m,k}x_j p_j\right]\\&\quad-(1-p_m)\left[x_m p_m+\sum_{j\neq m,k}x_j p_j\right]+2p_k(1-p_k)(x_m-x_k)\end{aligned}$$

$$= (p_m - p_k) \sum_{j \neq m,k} x_j p_j$$

$$+ [2p_k(1-p_k) - p_m(1-p_m)]\, x_m - p_k(1-p_k)\, x_k$$

$$\geqslant p_k(1-p_k)\, x_m \left[2 - \frac{x_k}{x_m} - \frac{p_m(1-p_m)}{p_k(1-p_k)} \right] \geqslant 0. \quad \blacksquare$$

Proof of Lemma 13. $\alpha_{1,n} = 1$ because an agent using the RSS always selects his most favored alternative. Similarly, $\beta_{1,n} = (n-1)/n$ because this is the probability that the other agent does not take the most preferred item of agent 2. With $n = 2$, agent 1 does not obtain his least favorite, so $\alpha_{2,2} = 0$, and agent 2 has an equal likelihood of getting and not getting his favorite; $\beta_{1,2} = \beta_{2,2} = \frac{1}{2}$.

Suppose there are n goods available. Number them according to agent 1's preferences, with the lowest number the most valuable. Agent 1 picks his favorite, good 1. What is the probability that agent 1 will eventually select his kth favorite? If agent 2 selects item k in his next pick, agent 1 will not get k, and this occurs with probability $1/(n-1)$. If agent 2 selects one of the items that agent 1 prefers to k, agent 1 will get k with probability $\alpha_{k-2,n-2}$, and this event occurs with probability $(k-2)/(n-1)$, bearing in mind that agent 1 has already selected one item. Finally, if agent 2 selects an item that agent 1 ranks below k, agent 1 obtains item k with probability $\alpha_{k-1,n-2}$. This gives the recursive formula for α.

Similarly, number the items according to agent 2's preferences, with low numbers corresponding to the most preferred items. If agent 1 selects agent 2's item k, then agent 2 does not receive k. If agent 1 selects an item that 2 prefers to k, which happens with probability $(k-1)/n$, then agent 2 becomes the next to select and gets k with probability $\alpha_{k-1,n-1}$. If agent 1 selects an item that is worse to agent 2 than item k, which happens with probability $(n-k)/n$, then agent 2 gets the item k with probability $\alpha_{k,n-1}$. This gives the formula for β.

Showing that, for n even, $\alpha_{k,n} = (n-k)/(n-1)$ is merely a matter of verifying the formula by induction because it holds for $n = 2$. The formula for β gives $\beta_{k,n} = (n-k)/(n-1)$ for n odd immediately, because $n-1$ is even.

For $n = 3$, $\alpha_{k,n} \geqslant (n-k)/(n-1)$, since $\alpha_{1,3} = 1$, $\alpha_{2,3} = \alpha_{3,3} = \frac{1}{2}$. By induction, for n odd, $\alpha_{k,n} \geqslant (n-k)/(n-1)$. But then, for n even,

$$\beta_{k,n} = \frac{k-1}{n} \alpha_{k-1,n-1} + \frac{n-k}{n} \alpha_{k,n-1} \geqslant \frac{k-1}{n} \frac{n-k}{n-2} + \frac{n-k}{n} \frac{n-1-k}{n-2}$$

$$= \frac{n-k}{n}, \qquad \text{as desired.} \quad \blacksquare$$

Proof of Theorem 14. The full information value associated with any one item is just the expectation of the maximum of two independent draws, or

$$\int_0^{x_H} x2F(x)f(x)\,dx = \int_0^{x_H} 1 - F(x)^2\,dx.$$

The expected maximum total value possible is n times this because of independence. The expected utility of agent 1 is

$$\begin{aligned}
\mathrm{EU}_1 &= \sum_{k=1}^{n-1} \alpha_{k,n} Ex_{(k)} \\
&= \sum_{k=1}^{n-1} \frac{n-k}{n-1} \int_0^{x_H} xn \binom{n-1}{k-1} (1-F(x))^{k-1} F(x)^{n-k} f(x)\,dx \\
&= n \int_0^{x_H} x \left(\sum_{k=1}^{n-1} \binom{n-2}{k-1} (1-F(x))^{k-1} F(x)^{n-k-1} \right) F(x) f(x)\,dx \\
&= n \int_0^{x_H} xF(x)f(x)\,dx = \frac{n}{2} \int_0^{x_H} 1 - F(x)^2\,dx.
\end{aligned}$$

Thus agent 1 obtains half the surplus. The expected surplus of player 2 is

$$\mathrm{EU}_2 = \sum_{k=1}^{n} \beta_{k,n} Ex_{(k)} \geqslant \sum_{k=1}^{n} \frac{n-k}{n} Ex_{(k)} = \frac{n-1}{n} \mathrm{EU}_1.$$

Thus, the total loss from the mechanism does not exceed $\frac{1}{2}\int_0^{x_H} 1 - F(x)^2\,dx \leqslant \frac{1}{2}x_H$. ∎

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