Maskin, E. 1992. "Auctions and Privatization." In *Privatization*, edited by H. Siebert. Institut für Weltwirtschaft an der Universität Kiel.
Milgrom, P. R., and R. J. Weber. 1982. "A Theory of Auctions and Competitive Bidding." *Econometrica* 50:1089–1122.
Milgrom, P. R., and C. Shannon. 1994. "Monotone Comparative Statics." *Econometrica* 62:157–180.
McMillan, J. 1994. "Selling Spectrum Rights." *Journal of Economic Perspectives* 8:145–162.
Pitchik, C., and A. Schotter. 1988. Perfect Equilibria in Budget-Constrained Sequential Auctions: An Experimental Study." *Rand Journal of Economics* 19:363–388.
Riley, J., and W. Samuelson. 1981. "Optimal Auctions." *American Economic Review* 71:381–392.
Sen, A. 1995. "On Seller Financing of Consumer Durables." Mimeo, University of Maryland.
Zeldes, S. 1989. "Consumption and Liquidity Constraints: An Empirical Investigation." *Journal of Political Economy* 97:1196–1216.

MATCHING AND EXPECTATIONS IN A MARKET WITH HETEROGENEOUS AGENTS

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ABSTRACT

Most existing models of decentralized markets either assume homogeneous agents or heterogeneous buyers and homogeneous sellers. In contrast, we examine the environment where both buyers and sellers are of continuums of types. With sequential random matching and bargaining, as might arise in a housing market, the expected utilities of all agents are endogenous. Thus, the types of agents that trade are determined by the equilibrium. For each buyer (seller), the set of seller (buyer) types for which trade occurs is an interval. Equilibrium prices vary with the matched agents' types. We compute the probability of trade and show that "middle" types have the highest probability of successful trade. Some comparative statics are conducted. The predictions are intuitive, for example, an increase in the probability of buyers of having a match would increase (decrease) the expected utilities of all types of buyers (sellers). We also establish a nonsteady state simulation model to study the effect of a uniform shock in demand (supply). A particularly interesting result is that transitory shocks may cause oscillations in prices, quantity traded, and the stock of types of agents in the market. Moreover, uniform shocks affect different types of agents differently.

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I. INTRODUCTION

A housing market has four characteristics that make it distinguishable from the traditionally modeled market. First, sellers have unique goods: no two houses are exactly alike. Second, the market is geographically dispersed, as buyers typically visit the sellers' houses. Third, after a single transaction, sellers tend to exit the market (or become buyers). Fourth, the transaction price is usually determined through bilateral negotiation or bargaining.[1] Although motivated by the housing market, the model developed is applicable to other markets with two-sided heterogeneity.

Such a market is sometimes modeled as a pair-wise random matching and bargaining process. The basic structure of this type of the model is as follows. There are two types of agents in the market, the buyer and the seller. Each agent intends to buy or sell exactly one unit of an indivisible commodity. In each period, each agent randomly matches with an agent from the opposite party. Once two agents meet, they bargain over the terms of the transaction. If an agreement is reached, a trade will take place and the agents involved in the trade will leave the market. If no agreement is reached, each will wait for a possible trade during the next period. We follow this basic framework and assume that all agents in the market have identical expectations on the division of the gain from trade in bargaining.

Most existing models of this type either assume homogeneous agents (Diamond and Maskin, 1979; Rubinstein and Wolinsky, 1985) or heterogeneous buyers and homogeneous sellers (McAfee, 1993; Peters, 1993; Wolinsky, 1988). In contrast, we assume that both buyers and sellers are heterogeneous and have distinguishable continuum of types.[2]

With sequential random matching and bargaining, the expected utilities of all agents are endogenous. Thus, the types of agents that trade with each other are endogenous. We show that the equilibrium matching exists and is unique (in steady state). At the equilibrium, we demonstrate that for each buyer (seller), the set of seller (buyer) types for which trade occurs is an interval. If a buyer (seller) matches with a seller (buyer) whose type is either too "low" or too "high," both would prefer to wait for the next period. This is intuitive: a buyer in the market for a given quality of house will not purchase one much worse, nor is the buyer willing to pay enough to buy one too much better. We show that the "middle" type agents have the broadest trade interval, and consequently the highest probability of successful trade. For each type of agents a best match exists. Lower type buyers prefer to match with lower type sellers; higher type buyers prefer to match with higher type sellers, and vice versa. Equilibrium prices vary with the matched agents' types. For a buyer, a higher quality house always means a higher price, but for a seller, a higher type buyer may bring him a lower price.

Comparative statics are conducted with respect to the probability of having a match, the common belief on the division of the gain from trade in bargaining, and the discount factor between two sequential periods. The predictions are perceptive. For example, an increase in the probability of buyers having a match would increase the expected utilities of all types of the buyers and decrease the expected utilities of all types of the sellers.

Away from the steady state, the dynamics of the system are enormously complicated, and we resort to simulation to analyze the effects of a temporary uniform surge in demand on the prices, the quantity traded of different types of houses, and the stock distributions of the types of the agents in the market. A particularly interesting feature of our finding is that a temporary increase in demand may cause oscillations in the prices, the quantity traded, and the stock of the types of the agents in the market. Moreover, a uniform shock in demand affects different types of agents differently. While the lower type buyers turns out to be the biggest losers, the lower type sellers will gain relatively the most, the middle type the second, and the higher type the least.

The paper contributes to the literature on price formation that explores the determinants of transactions prices in a decentralized setting. Much of the recent literature focuses on the choice of institution in determining price (e.g., McAfee, 1993; Peters, 1993). Should a seller post a price, hold an auction, or bargain with buyers? The literature on endogenous transaction methods grew out of an earlier literature in which the transaction institution was taken as exogenous (e.g., Diamond and Maskin, 1979; Rubinstein and Wolinsky, 1985 assume bilateral bargaining; Wolinsky, 1988 assumes sellers hold auctions). Due to the complexity of the analysis, the literature has focused on either identical agents, or one-sided asymmetries and usually identical sellers and differentiated buyers. In contrast, we examine two-sided differentiation, so that both buyers and sellers vary in type. While not permitting institutional choice in the present study, we view endogenizing the transaction institution as the logical next step.

The most closely related model to the present study is that of Baye and Cosimano (1990) which studies a matching market not unlike the present study. The major differences are that Baye and Cosimano study a one-shot framework, while we study a repeated framework, and Baye and Cosimano generalize to endogenize participation by all agents, where our agents lack a nonparticipation option. The repeated nature of the game studied here implies agents have an endogenous value of not trading.

The paper is organized as follows. In Section II the general dynamic model for a decentralized market with heterogeneous buyers and sellers is presented. In Section III the steady-state equilibrium model is analyzed and some comparative statics are conducted. In Section IV we introduce a nonsteady state simulation model and investigate the effects of a shock in demand. In Section V we present our conclusions.

II. THE MODEL

Consider a market in which each seller has a nondivisible good to sell and each buyer seeks to buy one and only one unit of the good. Both the buyers and the sellers are heterogeneous from a continuum of types. The sellers are classified by the values of their goods for sale. If a seller has a higher quality good for sale, then he is a higher type seller. The buyers are sorted by their ability to utilize houses. If a buyer can gain higher satisfaction from consuming a house, he is a higher type buyer. We use s to index the types of the sellers and b to index the types of the buyers, where $s \in [0,1]$ and $b \in [0,1]$.

Time is discrete $t = 1, 2, 3, \ldots$. In each period t the buyers and the sellers are randomly matched in pairs. There are three possible outcomes for an agent under such a match. First, he matches with no one. In this occasion, he has to wait for a possible match next period. Second, he does find a match but the gain from trade is negative. When this happens no trade will take place since at any price there is at least one agent who will be better off to wait for the next period. Third, the match results in nonnegative gain from trade. Only in this case, bargaining between the buyer and the seller on the terms of the transaction occurs, and a trade will take place. Once a trade is completed both the buyer and the seller involved in the trade leave the market. Next, a fraction of the remaining buyers and sellers is terminated exogenously,[3] and the rest of the agents wait for an opportunity next period. Subsequently, new buyers and sellers join the market and the process continues. It is assumed that the buyers and the sellers have the same discount factor over the value one period ahead.

We assume bargaining shares the gain from trade according to the generalized Nash bargaining solution, with the share θ_b (θ_s) of the gains from trade accruing to the buyer (seller). While this assumption is made for convenience, it is justified by Rubinstein's (1982) model. The outcome of Rubinstein and Wolinsky's (1985) model also comes in this form, where θ_b and θ_s depend on the ratio of buyers and sellers in the market. Note $\theta_b + \theta_s = 1$.

Assume that the utility of a type b buyer from consuming a type s house in all periods is bs.[4] By this assumption either the higher the quality of the house or the ability of the buyer to utilize a house, the higher the buyer's consumption value of the house will be. There are two sources of discounting in the model: pure time preference and the probability of termination. These two sources affect agents identically, and we denote the overall discount factor by δ. We use $U^t(b)$ and $V^t(s)$ to represent the expected utilities for a type b buyer and a type s seller in period t. By employing these notations, the gain from trade when a type b buyer meets with a type s seller can be expressed as

$$\Gamma^t(b,s) = bs - \delta U^{t+1}(b) - \delta V^{t+1}(s). \tag{1}$$

If a buyer matches with a seller but $\Gamma^t(b,s) < 0$, no trade will take place. If $\Gamma^t(b,s) \geq 0$, a trade will occur and the price will be

$$P^t(b,s) = \delta V^{t+1}(s) + \theta_s \Gamma^t(b,s). \tag{2}$$

To describe the expected utilities for all types of the buyers and the sellers, we define $f^t(s)$ and $g^t(b)$ as the density functions of the sellers' type and buyers' type. It is assumed that $f^t(s) > 0$ when $s \in (0,1)$ and $g^t(b) > 0$ when $b \in (0,1)$. We use p_b^t $(0 < p_b^t < 1)$ and p_s^t $(0 < p_s^t < 1)$ to denote respectively the probability of having a match for a buyer and a seller. Then, the expected utility of a type b buyer in each period t emerges as:

$$U^t(b) = (1 - p_b^t)\delta U^{t+1}(b) + p_b^t(1 - F^t(\alpha^t(b)))\delta U^{t+1}(b) + p_b^t \int_{\alpha^t(b)} (bs - P^t(b,s)) f^t(s)ds, \tag{3}$$

where $\alpha^t(b) = \{s \mid \Gamma^t(b,s) \geq 0\}$, and $F^t(\cdot)$ is the distribution function of the buyers' types. The first term of the right-hand side of equation (3) represents the possibility that a type b buyer may not match with a seller. The second term corresponds to the possibility that the buyer does have a match but that the gain from trade is negative. In these two occasions the buyer will either voluntarily or involuntarily wait for the next period with an expectation of $\delta U^{t+1}(b)$. The third term represents the possibility that the buyer matches with a seller and the gain from trade is nonnegative. In this case, the buyer expects a surplus of

$$\int_{\alpha^t(b)} (bs - P^t(b,s)) f^t(s) ds.$$

Similarly, the expected utility of a type s seller in period t is

$$V^t(s) = (1 - p_s^t)\delta V^{t+1}(s) + p_s^t(1 - G^t(\beta^t(s)))\delta V^{t+1}(s) + p_s^t \int_{\beta^t(s)} P^t(b,s) g^t(b) db, \tag{4}$$

where $\beta^t(s) = \{b \mid \Gamma^t(b,s) \geq 0\}$, and $G^t(\cdot)$ is the distribution function of the sellers' types.

Substituting the expression of the expected price into equations (3) and (4), the formula describing the expected utilities of the buyers and the sellers can be simplified to

$$U^t(b) = \delta U^{t+1}(b) + p_b^t \theta_b \int_{\alpha^t(b)} \Gamma^t(b,s) f^t(s) ds, \tag{5}$$

$$V^t(s) = \delta V^{t+1}(s) + p_s^t \theta_s \int_{\beta^t(s)} \Gamma^t(b,s) g^t(b) db. \tag{6}$$

In other words, the expected utility of an agent is the sum of his discounted expected utility next period and his expected average share of positive gain from trade during the current period.

Current expected utility depends on the discount factor, the probability of a match, the share the agent obtains from the gains from trade, and future expected utilities. Moreover, there are "interaction effects" in that current expected utility for buyers depends on the future expected utility of the other types of buyers, by way of the other buyers types affecting the gains of trade to sellers. In addition a given seller's house will have a stochastic price, since the price depends on the type of buyer with whom the seller is matched. All of these dependencies seem realistic.

III. STEADY STATE

When the inflows of both buyers and sellers are constant, the distributions of the types of the buyers and the sellers in the market will converge to a steady state distribution. Therefore, the probabilities of having a match, the prices, and the expected utilities will also converge. We investigate the properties of any steady state by dropping the time index t in the expressions of the expected utilities. This yields

$$U(b) = \delta U(b) + p_b \theta_b \int_0^1 \max\{0, \Gamma(b,s)\} f(s) ds$$

$$= \int_0^1 \max\{\delta U(b), \delta U(b) + p_b \theta_b \Gamma(b,s)\} f(s) ds. \tag{7}$$

$$V(s) = \delta V(s) + p_s \theta_s \int_0^1 \max\{0, \Gamma(b,s)\} g(b) db$$

$$= \int_0^1 \max\{\delta V(s), \delta V(s) + p_s \theta_s \Gamma(b,s)\} g(b) db. \tag{8}$$

We show that the expected utilities specified above for each type of buyers and sellers exist and are unique. Below, f and g represent density of the stock of buyers and sellers.

Proposition 1. Let $f(x)$ and $g(x)$ be smooth density functions that are defined on [0, 1]. Then for $0 \leq \delta < 1$, there exist unique continuous functions $U(y)$ and $V(y)$ on [0, 1] such that

$$U(y) = \int_0^1 \max\{\delta U(y), \delta U(y) + p_b \theta_b (xy - \delta U(y) - \delta V(x))\} f(x) dx, \tag{9}$$

$$V(y) = \int_0^1 \max\{\delta V(y), \delta V(y) + p_s\theta_s(xy - \delta U(x) - \delta V(y))\}g(x)dx. \quad (10)$$

Proofs are contained in the Appendix.

$\alpha(b)$, where $\alpha(b) = \{s \mid \Gamma(b,s) \geq 0\}$, is the set of sellers' types such that a match with a b type buyer creates a nonnegative gain from trade. Similarly, $\beta(s)$, where $\beta(s) = \{b \mid \Gamma(b,s) \geq 0\}$, is the set of buyers' types such that a match with an s type seller creates a nonnegative gain from trade. It is profitable for both the buyer and the seller to trade if and only if the gain from trade is nonnegative, so that these sets contain all the information we need to decide when a trade will occur and who have the highest probability of successful trade.

Proposition 2. $\alpha(b)$ and $\beta(s)$ are convex sets for all $b \in [0,1]$ and $s \in [0,1]$.

By Proposition 2, $\alpha(b)$ and $\beta(s)$ are convex sets for all $b \in [0,1]$ and $s \in [0,1]$. In other words, they are all intervals, possibly degenerate (either empty or a single point). For convenience, we call $\alpha(b)$ the trade interval for a b type buyer and $\beta(s)$ the trade interval for an s type seller. The following proposition indicates further that $\alpha(b)$ and $\beta(s)$ are nonempty intervals. Accordingly, we can express $\alpha(b)$ and $\beta(s)$ as $\alpha(b) = [\underline{\alpha}(b), \bar{\alpha}(b)]$ and $\beta(s) = [\underline{\beta}(s), \bar{\beta}(s)]$, where $\underline{\alpha}(b) = \min_s \alpha(b)$, $\bar{\alpha}(b) = \max_s \alpha(b)$, $\underline{\beta}(s) = \min_b \beta(s)$ and $\bar{\beta}(s) = \max_b \beta(s)$, are the boundaries of the trade intervals.

Proposition 3. (1) $\underline{\alpha}(0) = \bar{\alpha}(0) = 0$ and $\underline{\beta}(0) = \bar{\beta}(0) = 0$. (2) $\underline{\alpha}(b) < \bar{\alpha}(b)$ if $b > 0$ and $\underline{\beta}(s) < \bar{\beta}(s)$ if $s > 0$.

Proposition 3 indicates that the trade interval for a lowest type agent contains only one point. For an agent of the other type, the trade interval is nondegenerate. Its immediate consequence is that $U(0) = V(0) = 0$, and $U(b) > 0$, $V(s) > 0$ for all $b, s > 0$.

In particular, a highest type agent in the market has a nondegenerate trade interval. Furthermore, for a highest type agent, the upper bound of his trade interval corresponds to the highest type of agents in the opposite party. This is revealed in Proposition 4.

Proposition 4. $\bar{\alpha}(1) = 1$ and $\bar{\beta}(1) = 1$.

In the next proposition, we show a monotonic property of the lower and upper bounds of the trade intervals with respect to the agent's types.

Proposition 5. (1) When $b < \underline{\beta}(1)$, $\bar{\alpha}(b)$ is strictly increasing with respect to b and when $b \geq \underline{\beta}(1)$, $\bar{\alpha}(b) = 1$; when $s < \underline{\alpha}(1)$, $\bar{\beta}(s)$ is strictly increasing with respect to s and when $s \geq \underline{\alpha}(1)$, $\bar{\beta}(s) = 1$. (2) $\underline{\alpha}(b)$ is strictly increasing for all $b \in [0,1]$; $\underline{\beta}(s)$ is strictly increasing for all $s \in [0,1]$.

Proposition 5 states that the region of nonnegative gains from trade in (b,s) space is defined by two strictly increasing curves, $\underline{\alpha}(b)$ and $\bar{\alpha}(b)$ (or $\underline{\beta}(s)$ and $\bar{\beta}(s)$). Proposition 5 is illustrated in Figure 1, where $f(s) = 1$, $g(b) = 1$, $p_b = p_s = 0.5$, $\theta_b = \theta_s = 0.5$, and $\delta = 0.95$. Not surprisingly, whenever $\bar{\alpha}(b) < 1$, $\underline{\beta}(\bar{\alpha}(b)) = b$, and, similarly, $\bar{\beta}(\underline{\alpha}(b)) = b$.

Notice that the middle types have the broadest range of acceptable matches and that the lower types have narrower trade intervals than the middle types and the higher types by the conclusions of Propositions 3, 4, and 5. It implies that the middle type agents have the highest probability of successful trade. This is natural because the middle types can

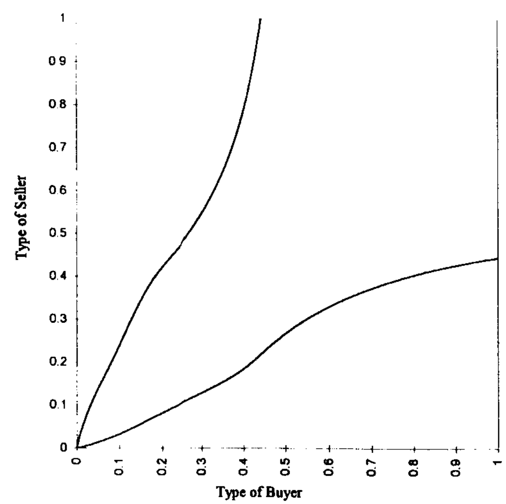

Figure 1. The Opportunity of Trade
($f(s) = 1$, $g(b) = 1$, $p_b = p_s = 0.5$, $\theta_b = \theta_s = 0.5$, and $\delta = 0.95$)

trade with either the lower types or the higher types while the lower and higher type's trading abilities are impaired by the truncations of the types at the extremes. The result that the lower types are restricted to fewer opportunities of making a trade indicates that people generally prefer higher type trading partners to lower type trading partners.

The heterogeneity of both the buyers and the sellers in the model makes it possible to compare the expected utilities of the different types of the agents.

Proposition 6. If $f(s) > 0$ for $s \in (0,1)$ and $g(b) > 0$ for $b \in (0,1)$, then $U(b)$ and $V(s)$ are strictly increasing when $b \in (0,1)$ and $s \in (0,1)$.

Notice that the results of Proposition 6 are commonly observable facts. If a seller owns a higher quality house for sale, he expects a higher price and therefore has a higher expected utility; if a seller owns a lower quality house for sale, he expects a lower price and therefore has a lower expected utility.

Proposition 7. $U(b)$ and $V(s)$ are strictly convex.

Proposition 7 shows that the expected utilities are convex. This is something of a surprise, for the usual intuition for convex utility in heterogeneous type models fails to hold in the present model. The usual intuition requires an agent to *mimic* another type and receive a payoff linearly related to that agent's type (e.g., a higher type buyer buying a lower type buyer's house). This mimicking is foreclosed, and this does not underlie the result. The result is likely to be sensitive to the multiplicative form of utility.

We now turn to answer the question if there is a best match for each agent in the market.

Proposition 8. Given b, $P(b,s)$ is strictly increasing in s; given s, $P(b,s)$ is maximized at b^* which satisfies

$$s - \delta \frac{\partial U(b)}{\partial b}\Big|_{b=b^*} = 0.$$

Proposition 8 shows that for a buyer of any type who would like to buy a higher quality house, he expects to pay more. But for a seller, the price of his house varies with the matched buyer's type. It is not necessarily true that a higher type buyer would offer a higher price

because of his higher expectation of next period. There is a certain type of buyer, which is specified by $s - \delta(\partial U(b)/\partial b)|_{b=b^*} = 0$, whose price would be the maximum for a type s house. This is surely the best type for a seller to match with.

Note that a buyer does not necessarily prefer to match with a lower type seller even though the price of a lower type house would be lower. In fact, the utility of a type b buyer buying a type s house is $bs - P(b,s)$, which is maximized at

$$b - \delta \frac{\partial V(s)}{\partial s}\Big|_{s=s^*} = 0.$$

Therefore, the type of the sellers that a type b buyer would prefer to match with is the type specified by

$$b - \delta \frac{\partial V(s)}{\partial s}\Big|_{s=s^*} = 0.$$

Figure 2, which is simulated under the same market condition as in Figure 1, displays the prices of a type 0.5 house for different type buyers.

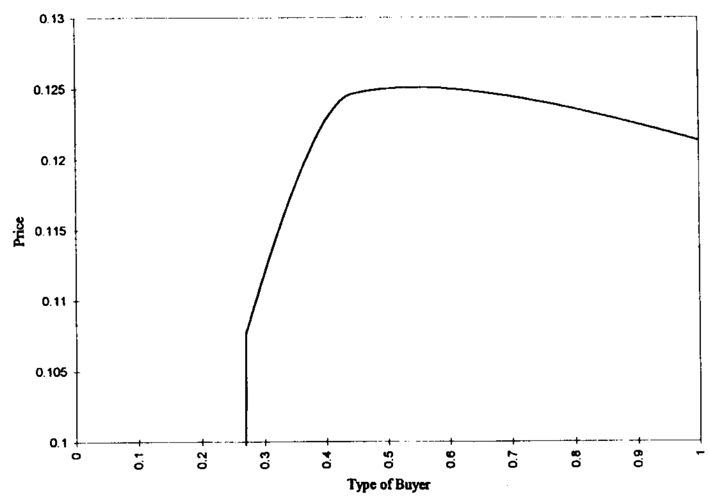

Figure 2. The Prices of a Type 0.50 Houses for Different Type Buyers ($f(s) = 1$, $g(b) = 1$, $p_b = p_s = 0.5$, $\theta_b = \theta_s = 0.5$ and $\delta = 0.95$)

By Figure 2, the most preferable buyer's type for a type 0.5 seller is approximately type 0.55. Since the buyer and the seller are symmetric in this case, a type 0.55 seller is also the best trading partner for a type 0.5 buyer.

Next, we explore the role that θ_b, θ_s, p_b, and p_s play in determining the expected utilities, the boundaries of the trade intervals, and the prices.

Proposition 9. Let $A = \theta_b, \theta_s, p_b$ or p_s, then $\partial\underline{\alpha}(b)/\partial A$ has the same sign as $\partial U(b)/\partial A$ and $\partial\underline{\beta}(s)/\partial A$ has the same sign as $\partial V(s)/\partial A$.

By Proposition 9, the effects of θ_b, θ_s, p_b, and p_s on the boundaries of the trade intervals completely depend on the signs of

$$\frac{\partial U(b)}{\partial \theta_i}, \frac{\partial U(b)}{\partial p_i}, \frac{\partial V(s)}{\partial \theta_i}, \text{ and } \frac{\partial V(s)}{\partial p_i} \ (i = b,s).$$

On the other hand,

$$\frac{\partial P(b,s)}{\partial A} = \delta(1-\theta_s)\frac{\partial V(s)}{\partial A} - \theta_s\delta\frac{\partial U(b)}{\partial A} + \Gamma(b,s)\frac{\partial \theta_s}{\partial A} \tag{11}$$

which means that the effects of θ_b, θ_s, p_b, and p_s on prices also rely on the sign of the derivatives. Therefore, the key to deciding the role that θ_b, θ_s, p_b, and p_s play in determining the expected utilities, the boundaries of the trade intervals, and the prices is to identify the signs of

$$\frac{\partial U(b)}{\partial \theta_i}, \frac{\partial U(b)}{\partial p_i}, \frac{\partial V(s)}{\partial \theta_i}, \text{ and } \frac{\partial V(s)}{\partial p_i} \ (i = b,s).$$

The results seem intuitively obvious, but the proof turns out to be difficult.

We simulated the equilibrium model with different values of the parameters and different distributions of agents' types (uniform and binomial distributions). The results show consistently that

$$\frac{\partial U(b)}{\partial \theta_b} > 0, \frac{\partial U(b)}{\partial \theta_s} < 0, \frac{\partial U(b)}{\partial p_b} > 0 \text{ and } \frac{\partial U(b)}{\partial p_s} < 0,$$

while

$$\frac{\partial V(s)}{\partial \theta_s} > 0, \frac{\partial V(s)}{\partial \theta_b} < 0, \frac{\partial V(s)}{\partial p_s} > 0, \text{ and } \frac{\partial V(s)}{\partial p_b} < 0.$$

When the probability of an agent's having a match, or his share of the gain from trade in bargaining increases, the expected utility of the agent will increase. Conversely, when the probability of his partner's having a match increases or when his partner's share of the gain from trade in bargaining increases, his expected utility will decrease.

The effects of θ_b, θ_s, p_b, and p_s on the boundaries of the trade intervals can be easily determined from the simulation results by using Proposition 9: for an agent of any type, the lower bounds of the trade intervals are positively related to the probability of his having a match, and his share of the gain from trade. It is negatively related to the probability of his partner's having a match, and his partner's share of the gain from trade. The same result holds for the upper bound of the trade interval when $\bar{\alpha}(b) < 1$ or $\bar{\beta}(s) < 1$. This is intuitive. For example, if the buyers and the sellers believe that the market has been changing in favor of buyers so that the bargaining power of the buyers in the market has increased, then the lower bound and upper bound of the trade intervals of all buyers will increase (when $\bar{\alpha}(b) = 1$, the upper bound will remain constant), and in contrast, the lower bound and upper bound of the trade intervals of all sellers will decrease.

As for the effect on the price, the simulation results imply (see equation (11))

$$\frac{\partial P(b,s)}{\partial A} \begin{cases} <0 & \text{when } A = \theta_b, p_b \\ >0 & \text{when } A = \theta_s, p_s \end{cases},$$

namely, the price is negatively related to the probability of the buyers finding a match, and the buyers' share of the gain from trade. It is positively related to the probability of the sellers finding a match, and the sellers' share of the gain from trade. If, for example, all agents in the market believe that the bargaining power of the buyers has increased, then the price will strictly decrease.

Recall that the discount factor is denoted δ. When $\delta = 0$, agents regard the utility of the next period as zero. In this case, an agent of any type would trade with anyone he meets. In other words, if there is a match, there is a trade. The expected utilities take a much simpler explicit form:

$$U(b) = p_b\theta_b b\int_0^1 sf(s)ds, \tag{12}$$

$$V(s) = p_s\theta_s s \int_0^1 bg(b)db. \tag{13}$$

The expected utilities are positive (except for the lowest type), strictly increasing and convex with respect to the type. They are also strictly monotonic increasing with respect to the probability of having a match and the share of the gain from trade. On the other hand, since $\theta_b + \theta_s = 1$, they are strictly decreasing with respect to the share of the gain from trade of the opposite party. However, the expected utilities are not necessarily decreasing with respect to the probability of the opposite party's having a match since it is not necessarily true that p_s is inversely related to p_b. The price is

$$P(b,s) = \theta_s bs \tag{14}$$

which is positively related to the types of both the buyer and the seller, and the share of the gain from trade believed to be received by the sellers. Notice that the best match for an agent of any type would be a highest type agent of the opposite party.

When $\delta = 1$, it can be shown that $U(b)$ and $V(s)$ are not unique. However, our computer simulation indicates that when δ approaches to one, the limits of $U(b)$ and $V(s)$ exist and their expressions are extremely simple:

$$U(b) = \frac{1}{2}b^2, \tag{15}$$

$$V(s) = \frac{1}{2}s^2. \tag{16}$$

We define them as the utilities at $\delta = 1$.

Notice that these expected utilities are independent of the probabilities of having a match. The reason for this is that there is no cost associated with waiting. As long as the probability of having a match in a single period is positive, the probability of having a match over time is equal to one. The expected utilities also do not depend on the sharing rule of the gain from trade in bargaining since the gain from trade when a type b buyer matches with a type s seller would be $\Gamma(b,s) = -(b-s)^2$, which will never be positive. A type b buyer would trade only with a type b seller and vise versa. The price would be

$$P(b,b) = \frac{1}{2}b^2. \tag{17}$$

One may view an increase in the discount factor δ as an increase in the frequency of matches, holding discounting of the future constant. There are two effects of an increase in δ. First, the future becomes more valuable, tending to increase utilities. Second, because waiting for an alternative match is less costly, the types of matches changes. For a high type agent, both of these effects are positive, and utilities increase with δ. The second effect is positive because high type agents become more likely to reject low type agents as δ increases, and thus the quality of their matches improves.

It is easily observed that the opposite is true for low type agents; that is, the second effect is both negative and can dominate the first. Consider an agent with type ε, small, but exceeding 0. With $\delta = 0$, this agent obtains $\frac{1}{2}\varepsilon\mu$, where μ is the mean of the other agents type. As $\delta \rightarrow 1$, this agent's utility converges to $\frac{1}{2}\varepsilon^2 < \frac{1}{2}\varepsilon\mu$ for all small ε. Thus, increases in the frequency of matching tend to benefit the high types at the expense of the low types.

IV. NONSTEADY STATE DYNAMICS

In this section, we introduce a nonsteady state simulation dynamic model based on the model presented in Section II to explore the effects of a uniform demand shock on the distributions of the types of the agents, the quantity traded, and the average prices for different quality goods.

As an approximation, we assume that there are $N + 1$ types for both the buyers and the sellers in the market, where N is a large positive integer. $b(i) = \frac{i}{N}$ and $s(j) = \frac{j}{N}$ are the type indexes for the buyers and the sellers, where $i, j = 0, 1, \ldots, N$. Let $B^t(i)$ be the number of type $b(i)$ buyers and $S^t(j)$ be the number of type $s(j)$ sellers in period t before matching. Then, the total number of buyers and sellers in the market $B^t = \Sigma_i B^t(i)$ and $S^t = \Sigma_j S^t(j)$, respectively.

By using B^t and S^t, the discrete distributions of the types of the agents in period t can be expressed as $f^t(j) = S^t(j)/S^t$ and $g^t(i) = B^t(i)/B^t$ ($i, j = 0, 1, \ldots, N$). We assume the probabilities of having a match for a buyer and a seller to be $p_b^t = S^t/(B^t + S^t)$ and $p_s^t = B^t/(B^t + S^t)$ respectively, which depend on the number of the buyers and the sellers in the current market. We use $\overline{B}^t(i)$ and $\overline{S}^t(j)$ to denote the number of the new buyers and the new sellers who enter the market in period t, and γ to denote the termination rate of the population.

Suppose that initially the market is in steady state: the inflow of buyers and sellers always matches with the outflow of buyers and sellers resulting from trade and termination, that is

$$\bar{B}^0(i) = B^0(i) - \left[1 - p_b^0 \sum_{j\in\alpha^0(i)} f^0(j)\right](1-\gamma)B^0(i), \tag{18}$$

$$\bar{S}^0(j) = S^0(j) - \left[1 - p_s^0 \sum_{i\in\beta^0(j)} g^0(i)\right](1-\gamma)S^0(j), \tag{19}$$

where, $\Gamma^0(i,j) = b(i)s(j) - \delta U^0(i) - \delta V^0(j)$, $\alpha^0(i) = \{j \mid \Gamma^0(i,j) \geq 0\}$ and $\beta^0(j) = \{i \mid \Gamma^0(i,j) \geq 0\}$; the distributions of the types of the buyers and the sellers remain to be $f^0(j) = S^0(j)/S^0$ and $g^0(i) = B^0(i)/B^0$ ($i, j = 0, 1, \ldots, N$); and the probabilities of having a match for a buyer and a seller are fixed at $p_b^0 = S^0/(B^0 + S^0)$ and $p_s^0 = B^0/(B^0 + S^0)$. The expected utilities of the agents and the prices of the good in steady state have the following expression:

$$U^0(i) = \delta U^0(i) + p_b^0\theta_b \sum_{j=1}^{N} \max\{0, \Gamma^0(i,j)\} f^0(j), \tag{20}$$

$$V^0(j) = \delta V^0(j) + p_s^0\theta_s \sum_{i=1}^{N} \max\{0, \Gamma^0(i,j)\} g^0(i), \tag{21}$$

$$AP^0(j) = \frac{\sum_{i\in\beta^0(j)} P^0(j)}{{}^{\#}\beta^0(j)} = \frac{\sum_{i\in\beta^0(j)} (\delta V^0(j) + \theta_s\Gamma^0(i,j))}{{}^{\#}\beta^0(j)}, \tag{22}$$

where ${}^{\#}\beta^0(j)$ is the number of the types of the buyers in the set $\beta^0(j)$.

Suppose now that there is a one-period uniform demand surge: the number of new buyers of all types entering the market in period one increases by 100 percent and in period two they return to the steady state level, while the inflow of the sellers continues to be the same as in steady state, that is

$$\bar{B}^1(i) = 2\bar{B}^0(i), \tag{23}$$

$$\bar{B}^{t+1}(i) = \bar{B}^0(i) \quad (t = 1, 2, \ldots), \tag{24}$$

while

$$\bar{S}^{t+1}(j) = \bar{S}^0(j) \quad (t = 0, 1, 2, \ldots). \tag{25}$$

Given the shock in demand in period one, the market is no longer in steady state equilibrium. It evolves afterwards according to the following dynamic equations (26)–(29) of the agents' types and utilities.

$$B^t(i) = \left[1 - p_b^{t-1} \sum_{j\in\alpha^{t-1}(i)} f^{t-1}(j)\right](1-\gamma)B^{t-1}(i) + \bar{B}^t(i), \tag{26}$$

$$S^t(j) = \left[1 - p_s^{t-1} \sum_{i\in\beta^{t-1}(j)} g^{t-1}(i)\right](1-\gamma)S^{t-1}(j) + \bar{S}^t(j), \tag{27}$$

$$U^t(i) = \delta U^{t+1}(i) + p_b^t\theta_b \sum_{j=1}^{N} \max\{0, \Gamma^t(i,j)\} f^t(j), \tag{28}$$

$$V^t(j) = \delta V^{t+1}(j) + p_s^t\theta_s \sum_{i=1}^{N} \max\{0, \Gamma^t(i,j)\} g^t(i), \tag{29}$$

$$(t = 1, 2, \ldots).$$

where, $\Gamma^t(i,j) = b(i)s(j) - \delta U^{t+1}(i) - \delta V^{t+1}(j)$, $\alpha^t(i) = \{j \mid \Gamma^t(i,j) \geq 0\}$ and $\beta^t(j) = \{i \mid \Gamma^t(i,j) \geq 0\}$.

Equations (26) and (27) indicate that the number of each type of the agents in period t is the summation of the number of the remaining agents after the trades and the termination in period $t - 1$, and the number of the new agents joining the market in period t. Equations (28) and (29) are just equations (5) and (6) in discrete form.

The average price for type $s(j)$ houses in period t can be calculated by equation (30).

$$AP^t(j) = \frac{\sum_{i\in\beta^t(j)} P^t(j)}{{}^{\#}\beta^t(j)} = \frac{\sum_{i\in\beta^t(j)} (\delta V^{t+1}(j) + \theta_s\Gamma^t(i,j))}{{}^{\#}\beta^t(j)}, \tag{30}$$

where, ${}^{\#}\beta^t(j)$ is the number of the types of the buyers in the set $\beta^t(j)$.

In our example, we let $N = 800$; that is, there are 801 types of buyers and 801 types of sellers in the market. The types of both buyers and sellers are uniformly distributed in initial steady state. Each type consists of 800 agents. We also assume that $\theta_b = \theta_s = 0.5$, $\delta = 0.95$, and $\gamma = 0.05$. Since $U^{t+1}(i)$ and $V^{t+1}(j)$ $(i,j = 0, 1, \ldots, N)$ are not available in period t, a two-step procedure is employed in simulation. In the first step, we substitute $U^{t-1}(i)$ and $V^{t-1}(j)$ for $U^{t+1}(i)$ and $V^{t+1}(j)$ in equations (26)–(29) to obtain the initial series of $U^t(i)$ and $V^t(j)$ $(i,j = 0, 1, \ldots, N$, and $t = 1, 2, \ldots, N)$. Notice that we calculate $U^t(i)$ and $V^t(j)$ just for N periods, since N period is long enough for the market to return to its initial steady state. In the second step, we apply equations (26)–(29) again to get new series of $U^t(i)$ and $V^t(j)$, using the initial series we generated in step one. This process continues until the limit of $U^t(i)$ and $V^t(j)$ are obtained. Figures 3 and 4 display the simulation results of the effect of the demand surge on the percentages of the type 0.05, 0.5, and 0.95 agents in the market. In Figure 3, the percentage of buyers is graphed, while Figure 4 provides the percentage of sellers of these types.

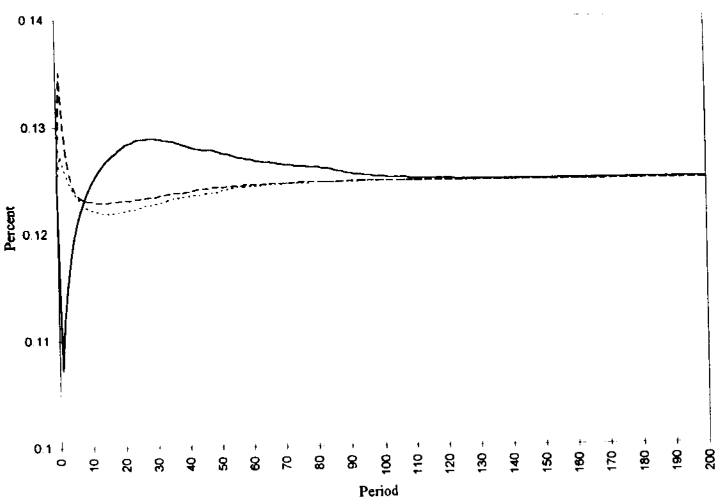

Figure 3. The Dynamics of the Percentages of the Type 0.05, 0.50, and 0.95 Buyers. Solid line, type 0.05; broken line: type 0.50; dotted line, type 0.95.

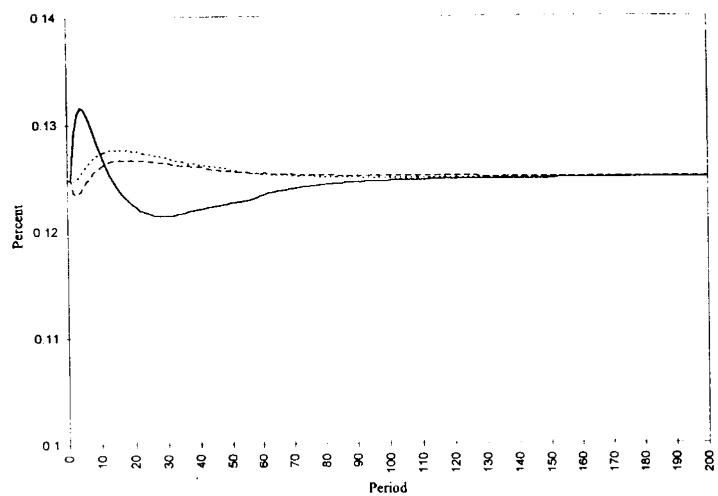

Figure 4. The Dynamics of the Percentages of the Type 0.05, 0.50, and 0.95 Sellers. Solid line, type 0.05; broken line: type 0.50; dotted line, type 0.95.

For both the buyer and the seller, the dynamic paths of the high type and the middle type agents are similar, while that of the low type agents takes a different course. The percentage of a middle or a high type of buyers increases when demand surges, then decreases to a level which is lower than its original, and then increases again and gradually returns to its initial level in steady state. As opposed to the middle and the high types, the percentage of a low type of buyer first decreases dramatically, then increases to a level which is higher than its original, and then decreases again and gradually returns to its initial level. The dynamic paths of the sellers look more like the mirror image of the dynamic paths of the buyers.

The patterns of these dynamic paths are closely related to the fact that the low type agents have less opportunity for making a trade than that of the middle type and the high type agents, as we pointed out in Section III (Figure 1). Because of the lower probability of having a trade for the low type agents, the number of the low type agents who leave or join the market each period in the steady state is less than either the middle or the high type agents. Therefore, when the number of all types of newcomer buyers doubles, the percentage of the low type buyers drops and the percentages of the middle and high type buyers surge.

Having more of the middle and the high type buyers in the market creates even more trading opportunities for the middle and the high type sellers. Consequently, in the second stage the quantity traded of the middle and the high type houses increases much faster than that of the low type houses, which causes the percentages of the middle and the high type sellers in the market to decrease dramatically while that of the low type sellers increases. When this happens, it diminishes the trading opportunities for the middle and the high type sellers and the trading opportunity for the low type sellers grows. Accordingly, the sale of the middle and the high type houses drops while that of the low type houses rises, which increases the numbers of the middle type and the high type sellers and decreases the number of the low type sellers in the market. There is a certain period in which the percentage of a middle or a high type sellers in the market is higher than that of the low type sellers.

In the third stage, the impact of the demand surge fades because of the partial termination of the population and the discount factor. The per-

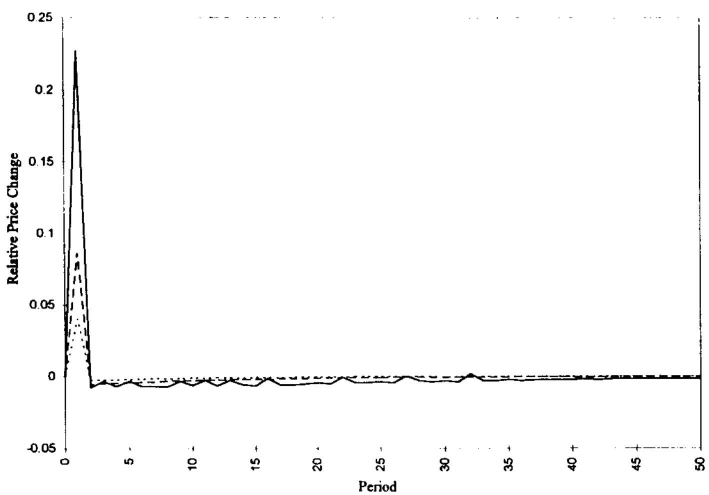

Figure 5. The Relative Price Changes for the Type 0.05, 0.50, and 0.95 Houses. Solid line, type 0.05; broken line: type 0.50; dotted line, type 0.95.

centages of all types of buyers and sellers in the market gradually return to their original steady state level.

Figure 5 displays the relative price changes of the type 0.05, 0.50, and 0.95 houses, after a demand surge.

The prices of the middle and high type houses jump immediately to the highest possible price when the demand increases, reflecting the perfect foresight of the agents. Then they gradually decrease and return to their original levels in steady state. Notice that the price of a 0.05, 0.5, and 0.95 type house increases by about 24 percent, 9 percent, and 4 percent, respectively, indicating that the lower the quality of a house, the greater its price fluctuates. It reveals that the lower type sellers who successfully traded in the period following a uniform temporary demand surge will benefit most, much more than that of the middle type and higher type, the middle type the second, and the higher type the least. Correspondingly, the lower type buyers are the biggest losers.

V. CONCLUSION

Based on the framework of pair-wise matching and bargaining, we established a model for a housing market in which both the buyers and the sellers are heterogeneous. In this model, the expected utilities, the trade intervals and the expected prices are characterized. The model provides an account of how price is determined in such a market and provide answers to questions like "who would trade with whom" or "when a trade would occur" and "is there a best match for an agent and who is the best match for an agent?" in such a market. The nonsteady state simulation model introduced in this paper could find general application in studying and predicting the effects of different types of demand or supply shocks in the market.

The novelty of the model is that it captures the complexity of the formation of expectations and price determination in a decentralized market by using a simple framework. Two interesting conjectures emerge from the simulations. First, overshooting in response to shocks is common; the rate of transactions may go above steady state, only to fall below, and return to steady state. Second, prices are more volatile for low-valued items. This conclusion is probably a consequence of the multiplicative nature of payoffs.

Although the model is presented in terms of housing market, it could be reinterpreted and applied to other decentralized markets. For example, we could apply it to a labor market by reinterpreting b as the index of the productivity of the firms and s as the index of the productivity of the workers.

APPENDIX

Proof of Proposition 1: Consider the set of continuous functions $C([0,1],R^2)$ which consists of all continuous maps of the closed interval $[0,1]$ into R^2. Obviously it is not empty.

For an arbitrary element $(U(y),V(y)) \in C([0,1],R^2)$, we define the norm as the usual superemum norm, that is,

$$\|(U(y),V(y))\| = \max\{\sup|U(y)|, \sup|V(y)|\}.$$

It is easy to verify that $C([0,1],R^2)$ is a Banach space with respect to the norm defined above.

Let $\mathbf{T}$ be a mapping from $C([0,1],R^2)$ to S, where

$$S = \{(U(y),V(y))|y \in [0,1]\} \text{ such that}$$

$$\mathbf{T}: (U(y),V(y)) \to$$

$$\left(\int_0^1 \max\{\delta U(y), \delta U(y) + p_b\theta_b(xy - \delta U(y) - \delta V(x))\} f(x)dx, \int_0^1 \max\{\delta V(y), \delta V(y) + p_s\theta_s(xy - \delta V(y) - \delta U(x))\} g(x)dx\right).$$

Obviously $\mathbf{T}$ is continuous, therefore $\mathbf{T}(U(y))$ and $\mathbf{T}(V(y))$ are continuous when both $U(y)$ and $V(y)$ are continuous. Hence, $\mathbf{T}$ is a mapping from $C([0,1],R^2)$ to $C([0,1],R^2)$. Now we show that $\mathbf{T}$ is a contraction mapping.

For all elements $((U_1(y), V_1(y)), (U_2(y), V_2(y))) \in C([0,1],R^2)$, let

$$\Gamma_1(y,x) = xy - \delta U_1(y) - \delta V_1(x) \text{ and } \Gamma_2(y,x) = xy - \delta U_2(y) - \delta V_2(x).$$

Then,

$$\|\mathbf{T}(U_1(y),V_1(y)) - \mathbf{T}(U_2(y),V_2(y))\|$$

$$= \max\{\sup_y \Big|\int_0^1 (\max\{\delta U_1(y), \delta U_1(y) + p_b\theta_b\Gamma_1(y,x)\}$$

$$- \max\{\delta U_2(y), \delta U_2(y) + p_b\theta_b\Gamma_2(y,x)\}) f(x)dx\Big|,$$

$$\sup_y \Big|\int_0^1 (\max\{\delta V_1(y), \delta V_1(y) + p_s\theta_s\Gamma_1(x,y)\}$$

$$- \max\{\delta V_2(y), \delta V_2(y) + p_s\theta_s\Gamma_2(x,y)\}) g(x)dx\Big|\}$$

$$\le \max\{\sup_y \int_0^1 \Big|\max\{\delta U_1(y), \delta U_1(y) + p_b\theta_b\Gamma_1(y,x)\}$$

$$- \max\{\delta U_2(y), \delta U_2(y) + p_b\theta_b\Gamma_2(y,x)\}\Big| f(x)dx,$$

$$\sup_y \int_0^1 \Big|\max\{\delta V_1(y), \delta V_1(y) + p_s\theta_s\Gamma_1(x,y)\}$$

$$- \max\{\delta V_2(y), \delta V_2(y) + p_s\theta_s\Gamma_2(x,y)\}\Big| g(x)dx\}.$$

For the first term in the last expression, if $\Gamma_1(y,x) \le 0$ and $\Gamma_2(y,x) \le 0$, then

$$\Delta = \Big|\max\{\delta U_1(y), \delta U_1(y) + p_b\theta_b\Gamma_1(y,x)\}$$

$$- \max\{\delta U_2(y), \delta U_2(y) + p_b\theta_b\Gamma_2(y,x)\}\Big|$$

$$= \delta|U_1(y) - U_2(y)|$$

$$\le \delta \max\{\sup_y |U_1(y) - U_2(y)|, \sup_y |V_1(y) - V_2(y)|\};$$

if $\Gamma_1(y,x) > 0$ and $\Gamma_2(y,x) > 0$, then

$$\Delta \le |(1 - p_b\theta_b)\delta(U_1(y) - U_2(y)) + p_b\theta_b\delta(V_2(y) - V_1(y))|$$

$$\le (1 - p_b\theta_b)\delta|U_1(y) - U_2(y)| + p_b\theta_b\delta|V_1(y) - V_2(y)|$$

$$\le \delta \max\{\sup_y |U_1(y) - U_2(y)|, \sup_y |V_1(y) - V_2(y)|\};$$

if $\Gamma_1(y,x) \le 0$ and $\Gamma_2(y,x) > 0$, then

$$\delta(U_2(y) + V_2(y)) < xy \le \delta(U_1(y) + V_1(y)),$$

hence

$$\delta U_1(y) - p_b\theta_b xy - (1 - p_b\theta_b)\delta U_2(y) + p_b\theta_b\delta V_2(x) \in$$

$$[\delta(U_2(y) - U_1(y)), (1 - p_b\theta_b)\delta(U_2(y) - U_1(y)) - p_b\theta_b\delta(V_2(y) - V_1(y))].$$

Since

$$\delta\,|U_2(y) - U_1(y)| \le \delta \max\{\sup_y |U_1(y) - U_2(y)|, \sup_y |V_1(y) - V_2(y)|\}$$

and

$$|(1 - p_b\theta_b)\delta(U_2(y) - U_1(y)) - p_b\theta_b\delta(V_2(y) - V_1(y))|$$
$$\le \delta \max\{\sup_y |U_1(y) - U_2(y)|, \sup_y |V_1(y) - V_2(y)|\},$$

thus

$$\Delta = |\delta U_1(y) - p_b\theta_b xy - (1 - p_b\theta_b)\delta U_2(y) + p_b\theta_b\delta V_2(x)|$$
$$\le \delta \max\{\sup_y |U_1(y) - U_2(y)|, \sup_y |V_1(y) - V_2(y)|\};$$

similarly, if $\Gamma_1(y, x) > 0$ and $\Gamma_2(y, x) \le 0$, then

$$\Delta = |p_b\theta_b xy + (1 - p_b\theta_b)\delta U_1(y) - p_b\theta_b\delta V_1(x) - \delta U_2(y)|$$
$$\le \delta \max\{\sup_y |U_1(y) - U_2(y)|, \sup_y |V_1(y) - V_2(y)|\}.$$

Therefore we conclude that

$$\Delta \le \delta \max\{\sup_y |U_1(y) - U_2(y)|, \sup_y |V_1(y) - V_2(y)|\}$$

always hold. Similarly, for the second term, we can prove that

$$|\max\{\delta V_1(y), \delta V_1(y) + p_s\theta_s\Gamma_1(x, y)\}$$
$$- \max\{\delta V_2(y), \delta V_2(y) + p_s\theta_s\Gamma_2(x, y)\}|$$
$$\le \delta \max\{\sup_y |U_1(y) - U_2(y)|, \sup_y |V_1(y) - V_2(y)|\}.$$

Therefore, when $0 < \delta < 1$,

$$\|\mathbf{T}(U_1(y), V_1(y)) - \mathbf{T}(U_2(y), V_2(y))\|$$
$$\le \delta \max\{\sup_y |U_1(y) - U_2(y)|, \sup_y |V_1(y) - V_2(y)|\}$$
$$= \delta\,\|(U_1(y), V_1(y)) - (U_2(y), V_2(y))\|$$
$$< \|(U_1(y), V_1(y)) - (U_2(y), V_2(y))\|.$$

That is, $\mathbf{T}$ is a contraction mapping. By Banach fixed-point theorem, there exists a unique pair of continuous functions $(U(y), V(y))$ on $[0,1]$ such that

$$U(y) = \int_0^1 \max\{\delta U(y), \delta U(y) + p_b\theta_b(xy - \delta U(y) - \delta V(x))\} f(x)dx,$$

$$V(y) = \int_0^1 \max\{\delta V(y), \delta V(y) + p_s\theta_s(xy - \delta U(x) - \delta V(y))\} g(x)dx.$$

Q.E.D.

Proof of Proposition 2: (1) We first prove that *U(b)* and *V(s)* are convex functions.

Let $b = \lambda b_1 + (1 - \lambda)b_2$, $\lambda \in [0,1]$, then

$$U(b_1) = \delta U(b_1) + p_b\theta_b \int_{\alpha(b_1)} \Gamma(b_1, s) f(s)ds,$$

hence,

$$U(b_1) = \frac{p_b\theta_b}{1 - \delta} \int_{\alpha(b_1)} \Gamma(b_1, s) f(s)ds$$
$$= \frac{p_b\theta_b}{1 - \delta} \max_{A \in R^1} \int_A \Gamma(b_1, s) f(s)ds$$
$$\ge \frac{p_b\theta_b}{1 - \delta} \int_{\alpha(b)} \Gamma(b_1, s) f(s)ds.$$

Similarly,

$$U(b_2) \geq \frac{p_b\theta_b}{1-\delta} \int_{\alpha(b)} \Gamma(b_2,s) f(s)ds.$$

Consequently,

$$((1-\delta) + p_b\theta_b\delta F(\alpha(b)))U(b_1) \geq p_b\theta_b \int_{\alpha(b)} (b_1 s - \delta V(s)) f(s)ds,$$

$$((1-\delta) + p_b\theta_b\delta F(\alpha(b)))U(b_2) \geq p_b\theta_b \int_{\alpha(b)} (b_2 s - \delta V(s)) f(s)ds$$

so that

$$((1-\delta) + p_b\theta_b\delta F(\alpha(b)))(\lambda U(b_1) + (1-\lambda)U(b_2))$$

$$\geq p_b\theta_b \int_{\alpha(b)} (bs - \delta V(s)) f(s)ds$$

$$= p_b\theta_b \int_{\alpha(b)} \Gamma(b,s) f(s)ds + p_b\theta_b \int_{\alpha(b)} \delta U(b) f(s)ds$$

$$= (1-\delta)U(b) + p_b\theta_b\delta F(\alpha(b))U(b)$$

$$= ((1-\delta) + p_b\theta_b\delta F(\alpha(b)))U(b),$$

hence $\lambda U(b_1) + (1-\lambda)U(b_2) \geq U(b)$. Therefore *U(b)* is a convex function. Similarly, it can be proved that *V(s)* is also a convex function. (2) *a(b)* is convex if *V(s)* is convex; $\beta(s)$ is convex if *U(b)* is convex. $\forall\, s_1, s_2 \in \alpha(b) = \{s \mid \Gamma(b,s) \geq 0\}$, by definition

$$bs_1 - \delta U(b) - \delta V(s_1) \geq 0,$$

$$bs_2 - \delta U(b) - \delta V(s_2) \geq 0.$$

Let $s = \lambda s_1 + (1-\lambda)s_2$, $\lambda \in [0,1]$, then

$$bs - \delta U(b) - \delta V(s)$$

$$= b(\lambda s_1 + (1-\lambda)s_2) - \delta U(b)(\lambda + (1-\lambda)) - \delta V(\lambda s_1 + (1-\lambda)s_2)$$

$$= \lambda(bs_1 - \delta U(b) - \delta V(s_1)) + (1-\lambda)(bs_2 - \delta U(b) - \delta V(s_2))$$

$$+ \delta(V(s_1) + (1-\lambda)V(s_2) - V(\lambda s_1 + (1-\lambda)s_2))$$

$$\geq \delta(V(s_1) + (1-\lambda)V(s_2) - V(\lambda s_1 + (1-\lambda)s_2)).$$

When *V(s)* is convex, $bs - \delta U(b) - \delta V(s) \geq 0$, hence $\alpha(b)$ is convex.

Similarly, when *U(b)* is convex $\beta(s)$ is convex. By (1) and (2), $\alpha(b)$ and $\beta(s)$ are convex for all $b \in [0,1]$ and $s \in [0,1]$ Q.E.D.

Proof of Proposition 3: (1) First we prove that for all $b \in [0,1]$ and $s \in [0,1]$,

$$\alpha(b) = \{s \mid \Gamma(b,s) \geq 0\} \neq \varnothing \text{ and } \beta(s) = \{b \mid \Gamma(b,s) \geq 0\} \neq \varnothing.$$

$$\forall\, b \in [0,1],$$

$$U(b) = \frac{p_b\theta_b}{1-\delta} \int_{\alpha(b)} \Gamma(b,s)\, f(s)ds$$

$$= \frac{p_b\theta_b}{1-\delta} \int_{\alpha(b)} (bs - \delta V(s)) f(s)ds - \frac{p_b\theta_b\delta}{1-\delta} F(\alpha(b))U(b).$$

It can be expressed as

$$U(b) = \int_{\alpha(b)} \frac{p_b\theta_b(bs - \delta V(s))}{(1-\delta) + p_b\theta_b\delta F(\alpha(b))} f(s)ds,$$

where $F(\cdot)$ is the distribution function corresponding to the density function $f(\cdot)$. When $b = 0$,

$$U(0) = \int_{\alpha(0)} \frac{p_b\theta_b(-\delta V(s))}{(1-\delta) + p_b\theta_b\delta F(\alpha(0))} f(s)ds.$$

Since both *U(b)* and *V(s)* are nonnegative for all *b* and *s*, $U(0) = 0$. Similarly, $V(0) = 0$.

For all *b* such that $U(b) = 0$, $\alpha(b) \neq \varnothing$ because $0 \in \alpha(b) = \{s \mid bs - \delta V(s) - \delta U(b) \geq 0\}$. For *b* satisfying $U(b) \neq 0$, by the definition of $U(b)$ we also have $\alpha(b) \neq \varnothing$. Hence for all $b \in [0,1]$, $\alpha(b) \neq \varnothing$. Similarly for all $s \in [0,1]$, $\beta(s) \neq \varnothing$.

By Proposition 2, $\alpha(b)$ and $\beta(s)$ are convex sets. Therefore, both $\alpha(b)$ and $\beta(s)$ are nonempty intervals, denoted as $\alpha(b) = [\underline{\alpha}(b), \overline{\alpha}(b)]$ and $\beta(s) = [\underline{\beta}(s), \overline{\beta}(s)]$, where

$$\underline{\alpha}(b) = \min_s \alpha(b),\ \overline{\alpha}(b) = \max_s \alpha(b),$$

$$\underline{\beta}(s) = \min_b \beta(s) \text{ and } \overline{\beta}(s) = \max_b \beta(s).$$

Note that $\underline{\alpha}(b)$, $\overline{\alpha}(b)$, $\underline{\beta}(s)$ and $\overline{\beta}(s)$ satisfy the following equations at the neighborhood of zero because of the continuity of the function $\Gamma(b,s)$:

$$s\underline{\beta}(s) - \delta V(s) - \delta U(\underline{\beta}(s)) = 0,$$

$$s\overline{\beta}(s) - \delta V(s) - \delta U(\overline{\beta}(s)) = 0,$$

$$b\underline{\alpha}(b) - \delta V(\underline{\alpha}(b)) - \delta U(b) = 0,$$

$$b\overline{\alpha}(b) - \delta V(\overline{\alpha}(b)) - \delta U(b) = 0.$$

By using $\underline{\alpha}(b)$, $\overline{\alpha}(b)$, $\underline{\beta}(s)$ and $\overline{\beta}(s)$, the expected utilities of the buyers and the sellers can be expressed as

$$U(b) = \frac{p_b\theta_b}{1-\delta}\int_{\underline{\alpha}(b)}^{\overline{\alpha}(b)} (bs - \delta V(s) - \delta U(b))f(s)ds,$$

$$V(s) = \frac{p_s\theta_s}{1-\delta}\int_{\underline{\beta}(s)}^{\overline{\beta}(s)} (bs - \delta V(s) - \delta U(b))g(b)db.$$

Take derivative with respect to b for the first equation, s for the second equation, we get

$$\frac{dU(b)}{db} = \int_{\underline{\alpha}(b)}^{\overline{\alpha}(b)} \frac{p_b\theta_b s}{(1-\delta) + p_b\theta_b\delta(F(\overline{\alpha}(b)) - F(\underline{\alpha}(b)))} f(s)ds,$$

$$\frac{dV(s)}{ds} = \int_{\underline{\beta}(s)}^{\overline{\beta}(s)} \frac{p_s\theta_s b}{(1-\delta) + p_s\theta_s\delta(G(\overline{\beta}(s)) - G(\underline{\beta}(s)))} g(b)db.$$

Now we prove that $\alpha(0) = \{0\}$ and $\beta(0) = \{0\}$. Obviously, $\underline{\alpha}(0) = 0$ and $\underline{\beta}(0) = 0$. We claim that $\overline{\alpha}(0)$ and $\overline{\beta}(0)$ are also equal to zero. Otherwise, suppose, for example, $\overline{\alpha}(0) > 0$. Then, $V(\overline{\alpha}(0)) = 0$ since $U(0) = 0$. On the other hand, by definition,

$$V(\overline{\alpha}(0)) = \int_{\beta(\overline{\alpha}(0))} \frac{p_s\theta_s(b\overline{\alpha}(0) - \delta V(\overline{\alpha}(0)) - \delta U(b))}{1-\delta} g(b)db$$

$$= \int_{\{b \mid b\overline{\alpha}(0) - \delta U(b) \geq 0\}} \frac{p_s\theta_s(b\overline{\alpha}(0) - \delta U(b))}{1-\delta} g(b)db.$$

Let $y_1 = b\overline{\alpha}(0)$ and $y_2 = \delta U(b)$, then $\lim_{b\to 0} dy_1/db = \overline{\alpha}(0) > 0$ and in the neighborhood of zero,

$$\frac{dy_2}{db} = \delta\frac{dU(b)}{db}$$

$$= \frac{p_b\theta_b\delta\int_{\underline{\alpha}(b)}^{\overline{\alpha}(b)} sf(s)ds}{(1-\delta) + p_b\theta_b\delta(F(\overline{\alpha}(b)) - F(\underline{\alpha}(b)))}$$

$$\leq \frac{p_b\theta_b\delta\overline{\alpha}(b)(F(\overline{\alpha}(b)) - F(\underline{\alpha}(b)))}{(1-\delta) + p_b\theta_b\delta(F(\overline{\alpha}(b)) - F(\underline{\alpha}(b)))},$$

$$\lim_{b\to 0}\frac{dy_2}{db} \leq \lim_{b\to 0}\frac{p_b\theta_b\delta\overline{\alpha}(b)(F(\overline{\alpha}(b)) - F(\underline{\alpha}(b)))}{(1-\delta) + p_b\theta_b\delta(F(\overline{\alpha}(b)) - F(\underline{\alpha}(b)))}$$

$$= \frac{p_b\theta_b\delta\overline{\alpha}(0)(F(\overline{\alpha}(0)) - F(\underline{\alpha}(0)))}{(1-\delta) + p_b\theta_b\delta(F(\overline{\alpha}(0)) - F(\underline{\alpha}(0)))} < \overline{\alpha}(0).$$

Hence, $\lim_{b\to 0} dy_1/db > \lim_{b\to 0} dy_2/db$. But at $b = 0$, $y_1 = y_2 = 0$, by continuity of the functions, there exists $b' > 0$ such that for all $b \in (0,b')$, $b\overline{\alpha}(0) > \delta U(b)$. Hence

$$V(\bar{\alpha}(0)) = \int_{\{b \mid b\bar{\alpha}(0) - \delta U(b) \geq 0\}} \frac{p_s\theta_s(b\bar{\alpha}(0) - \delta U(b))}{1-\delta} g(b)db > 0.$$

This is a contradiction. Hence, $\bar{\alpha}(0) = 0$. Similarly $\bar{\beta}(0) = 0$. Therefore

$$\alpha(0) = \{0\} \text{ and } \beta(0) = \{0\}.$$

(2) By (1) we have

$$\{0\} = \alpha(0) = \{s \mid -\delta V(s) \geq 0\} = \{s \mid V(s) \leq 0\}$$

and

$$\{0\} = \beta(0) = \{b \mid -\delta U(b) \geq 0\} = \{b \mid U(b) \leq 0\}$$

hence, for all $b \neq 0$ and $s \neq 0$, $U(b) > 0$ and $V(s) > 0$. By the definitions of $U(b)$ and $V(s)$, we conclude that for all $b \neq 0$ and all $s \neq 0$ $\bar{\alpha}(b) > \underline{\alpha}(b)$ and $\bar{\beta}(s) > \underline{\beta}(s)$. Q.E.D.

Proof of Proposition 4: We only prove the first equation. The proof of the second equation is basically the same.

We first prove that for all $b \in (0,1)$,

$$\frac{dU(b)}{ds} = \int_{\underline{\alpha}(b)}^{\bar{\alpha}(b)} \frac{p_b\theta_b s}{(1-\delta) + p_b\theta_b\delta(F(\bar{\alpha}(b)) - F(\underline{\alpha}(b)))} f(s)ds.$$

For $0 < b < \underline{\beta}(1)$, or $b > \bar{\beta}(1)$, it must be true that $\bar{\alpha}(1) < 1$. By the continuity of the function of $\Gamma(b,s)$, we have $b\underline{\alpha}(b) - \delta V(\underline{\alpha}(b)) - \delta U(b) = 0$ and $b\bar{\alpha}(b) - \delta V(\bar{\alpha}(b)) - \delta U(b) = 0$. Hence, the above equation holds.

For $\underline{\beta}(1) \leq b \leq \bar{\beta}(1)$, $b\underline{\alpha}(b) - \delta V(\underline{\alpha}(b)) - \delta U(b) = 0$ is still valid, but, $b\bar{\alpha}(b) - \delta V(\bar{\alpha}(b)) - \delta U(b) \neq 0$ since $\bar{\alpha}(b) = 1$. However, when $\bar{\alpha}(b) = 1$, $d\bar{\alpha}(b)/db = 0$, therefore, the above expression of the derivative for the utility still holds.

Suppose Proposition 4 is not true, that is, $\bar{\beta}(1) < 1$, where $\bar{\beta}(1) - \delta V(1) - \delta U(\bar{\beta}(1)) \geq 0$, then there exists $b = \bar{\beta}(1) + \Delta b$ $(\Delta b > 0)$ such that $b - \delta V(1) - \delta U(b) < 0$. Hence,

$$(b - \bar{\beta}(1)) - \delta(U(b) - U(\bar{\beta}(1))) < 0$$

or

$$\frac{U(\bar{\beta}(1) + \Delta b) - U(\bar{\beta}(1))}{\Delta b} > \frac{1}{\delta}.$$

Therefore,

$$\frac{dU(b)}{db}\Big|_{b=\bar{\beta}(1)} \geq \frac{1}{\delta}.$$

But, for all $b \in (0,1)$,

$$\frac{dU(b)}{db} = \int_{\underline{\alpha}(b)}^{\bar{\alpha}(b)} \frac{p_b\theta_b s}{(1-\delta) + p_b\theta_b\delta(F(\bar{\alpha}(b)) - F(\underline{\alpha}(b)))} f(s)ds$$

$$\leq \frac{1}{\delta}\left(\frac{p_b\theta_b\delta(F(\bar{\alpha}(b)) - F(\underline{\alpha}(b)))}{(1-\delta) + p_b\theta_b\delta(F(\bar{\alpha}(b)) - F(\underline{\alpha}(b)))}\right)$$

$$< \frac{1}{\delta}.$$

This is a contradiction. Therefore $\bar{\beta}(1) = 1$. Q.E.D.

Proof of Proposition 5: First we prove that $U(b)$ is strictly convex when $b < \underline{\beta}(1)$.

$\forall\, b_1, b_2 < \underline{\beta}(1)$, where, $b_1 \neq b_2$,

$$\alpha(b_1) = \{s \mid b_1 s - \delta V(s) \geq \delta U(b_1)\} = [\underline{\alpha}(b_1), \bar{\alpha}(b_1)],$$

$$\alpha(b_2) = \{s \mid b_2 s - \delta V(s) \geq \delta U(b_2)\} = [\underline{\alpha}(b_2), \bar{\alpha}(b_2)],$$

then $\alpha(b_1) \neq \alpha(b_2)$. Otherwise, suppose $b_1 > b_2$ but $\underline{\alpha}(b_1) = \underline{\alpha}(b_2)$ and $\bar{\alpha}(b_1) = \bar{\alpha}(b_2)$. Since $b_1\underline{\alpha}(b_1) - \delta V(\underline{\alpha}(b_1)) = \delta U(b_1)$ and $b_1\bar{\alpha}(b_1) - \delta V(\bar{\alpha}(b_1)) = \delta U(b_1)$, we have

$$b_1\underline{\alpha}(b_1) - \delta V(\underline{\alpha}(b_1)) = b_1\bar{\alpha}(b_1) - \delta V(\bar{\alpha}(b_1));$$

similarly

$$b_2\underline{\alpha}(b_2) - \delta V(\underline{\alpha}(b_2)) = b_2\bar{\alpha}(b_2) - \delta V(\bar{\alpha}(b_2)).$$

Subtracting one equation from the other we get

$$(b_1 - b_2)\underline{\alpha}(b_1) = (b_1 - b_2)\bar{\alpha}(b_1),$$

but $b_1 \neq b_2$ so that $\underline{\alpha}(b_1) = \bar{\alpha}(b_1)$. This is a contradiction since $\underline{\alpha}(b_1) < \bar{\alpha}(b_1)$. Therefore $\alpha(b_1) \neq \alpha(b_2)$. Let $b = \lambda b_1 + (1 - \lambda)b_2$ and $\lambda \in (0,1)$. Applying the same arguments to (b,b_1) and (b,b_2), we obtain that $\alpha(b) \neq \alpha(b_1)$ and $\alpha(b) \neq \alpha(b_2)$. Thus,

$$U(b_1) = \frac{p_b\theta_b}{1-\delta}\int_{\alpha(b_1)} \Gamma(b_1,s)f(s)ds$$

$$\geq \frac{p_b\theta_b}{1-\delta}\int_{\alpha(b)\cup(\alpha(b_1)-\alpha(b))} \Gamma(b_1,s)f(s)ds$$

$$\geq \frac{p_b\theta_b}{1-\delta}\int_{\alpha(b)} \Gamma(b_1,s)f(s)ds$$

$$+ \frac{p_b\theta_b}{1-\delta}\int_{(\alpha(b_1)-\alpha(b))} \Gamma(b_1,s)f(s)ds$$

$$> \frac{p_b\theta_b}{1-\delta}\int_{\alpha(b)} \Gamma(b_1,s)f(s)ds,$$

and

$$U(b_2) \geq \frac{p_b\theta_b}{1-\delta}\int_{\alpha(b)} \Gamma(b_2,s)f(s)ds.$$

Applying the same arguments in the proof of Proposition 2, we obtain that *U(b)* is strictly convex when $b < \underline{\beta}(1)$. Similarly *V(s)* is also strictly convex when $s < \underline{\alpha}(1)$.

Next, we prove that $\bar{\alpha}(b)$ is strictly increasing with respect to b when $b < \underline{\beta}(1)$ and $\bar{\alpha}(b) = 1$ when $b \geq \underline{\beta}(1)$.

$\forall\, b', b'' < \underline{\beta}(1)$. Suppose $b' > b''$, but $\bar{\alpha}(b') < \bar{\alpha}(b'')$. It is easy to see that $\bar{\alpha}(b') < \bar{\alpha}(b'') < 1 = \bar{\alpha}(\underline{\beta}(1))$. Since $\bar{\alpha}(b)$ is continuous,

$\exists\, b \in (b',\underline{\beta}(1))$ s.t. $\bar{\alpha}(b) = \bar{\alpha}(b'')$. Obviously, $b'',b \in \beta(\bar{\alpha}(b''))$ but $b' \notin \beta(\bar{\alpha}(b''))$, while $b > b' > b''$, hence $\beta(\bar{\alpha}(b''))$ is not a convex set. This is a contradiction. Therefore $\bar{\alpha}(b') \geq \bar{\alpha}(b'')$. Consequently, $d\bar{\alpha}(b)/db \geq 0$. We claim that $d\bar{\alpha}(b)/db \neq 0$. Otherwise, by $b\bar{\alpha}(b) - \delta V(\bar{\alpha}(b)) - \delta U(b) = 0$, we have

$$\frac{d\bar{\alpha}(b)}{db} = \frac{\delta\dfrac{dU(b)}{db} - \bar{\alpha}(b)}{b - \delta\dfrac{V(\bar{\alpha}(b))}{ds}} = 0$$

which means that $\bar{\alpha}(b) = \delta\, dU(b)/db$. Take derivative with respect to b, we get

$$\frac{d\bar{\alpha}(b)}{db} = \delta\frac{d^2U(b)}{db^2}.$$

Since the expected utility is strictly convex when $b < \underline{\beta}(1)$, the right-hand side of the equation is positive, therefore, $d\bar{\alpha}(b)/db > 0$. This is a contradiction. Therefore $\bar{\alpha}(b)$ is strictly increasing with respect to b when $b < \underline{\beta}(1)$. By Proposition 4, $\bar{\alpha}(1) = 1$. On the other hand, $\bar{\alpha}(\underline{\beta}(1)) = 1$, while $\beta(1)$ is a convex set. Therefore $\bar{\alpha}(b) = 1$ when $b \geq \underline{\beta}(1)$.

Similarly, $\bar{\beta}(s)$ is strictly increasing with respect to s when $s < \underline{\alpha}(1)$, and $\bar{\beta}(s) = 1$ when $b \geq \underline{\beta}(1)$.

Now we prove that $\underline{\alpha}(b)$ is strictly increasing with respect to b for all $b \in [0,1]$ and $\underline{\beta}(s)$ is strictly increase with respect to s for all $s \in [0,1]$. But these are obvious since $\underline{\alpha}(b)$ is the inverse function of $\bar{\beta}(s)$ and $\underline{\beta}(s)$ is the inverse function of $\bar{\alpha}(b)$. Q.E.D.

Proof of Proposition 6: We showed in the proof of Proposition 4 that for all $b \in (0,1)$,

$$\frac{dU(b)}{db}\int_{\underline{\alpha}(b)}^{\bar{\alpha}(b)} \frac{p_b\theta_b s}{(1-\delta) + p_b\theta_b\delta(F(\bar{\alpha}(b)) - F(\underline{\alpha}(b)))} f(s)ds$$

Since $f(s) > 0$ for $s \in [0,1]$, $p_b > 0$, $\theta_b > 0$ and $\bar{\alpha}(b) > \underline{\alpha}(b)$, when $b \in (0,1)$, we have $dU(b)/db > 0$, that is, *U(b)* is strictly increasing when $b \in (0,1)$. Similarly *V(s)* is also strictly increasing when $s \in (0,1)$. Q.E.D.

Proof of Proposition 7: By Proposition 5, $\forall\, b_1, b_2 \in [0,1]$, if $b_1 \neq b_2$ then $\alpha(b_1) \neq \alpha(b_2)$; $\forall\, s_1, s_2 \in (0,1)$, if $s_1 \neq s_2$, then $\beta(s_1) \neq \beta(s_2)$. Following the steps in the first part of the proof of Proposition 5, it is easy to show that *U(b)* and *V(s)* are strictly convex. Q.E.D.

Proof of Proposition 8: Given *nb*, since

$$\frac{\partial P(b,s)}{\partial s} = \frac{\partial}{\partial s}(\delta V(s) + \theta_s(bs - \delta U(b) - \delta V(s)))$$

$$= \theta_s b + (1-\theta_s)\delta\frac{\partial V(s)}{\partial s} > 0,$$

hence, *P(b,s)* is strictly increasing in *s* for given *nb*.

On the other hand, since

$$\frac{\partial P(b,s)}{\partial b}\Big|_{b=b^*} = \frac{\partial}{\partial b}(\delta V(s) + \theta_s(bs - \delta U(b) - \delta V(s)))\Big|_{b=b^*}$$

$$= \theta_s\Big(s - \delta\frac{\partial U(b)}{\partial b}\Big|_{b=b^*}\Big) = 0,$$

and

$$\frac{\partial^2 P(b,s)}{\partial b^2} = -\theta_s\delta\frac{\partial^2 U(b)}{\partial b^2} < 0,$$

hence, *P(b,s)* is maximized at b^* which satisfies

$$s - \delta\frac{\partial U(b)}{\partial b}\Big|_{b=b^*} = 0$$

Q.E.D.

Proof of Proposition 9: We know that $b\underline{\alpha}(b) - \delta V(\underline{\alpha}(b)) - \delta U(b) = 0$ for $b \in [0,1]$ and $s\underline{\beta}(s) - \delta V(s) - \delta U(\underline{\beta}(s)) = 0$ for $s \in [0,1]$. Take derivative with respect to *A*, we get

$$b\frac{\partial\underline{\alpha}(b)}{\partial A} - \delta\frac{\partial V(\underline{\alpha}(b))}{\partial s}\frac{\partial\underline{\alpha}(b)}{\partial A} - \delta\frac{\partial U(b)}{\partial A} = 0,$$

$$\frac{\partial\underline{\beta}(s)}{\partial A}s - \delta\frac{\partial V(s)}{\partial A} - \delta\frac{\partial U(\underline{\beta}(s))}{\partial b}\frac{\partial\underline{\beta}(s)}{\partial A} = 0,$$

or

$$\frac{\partial\underline{\alpha}(b)}{\partial A} = \frac{\delta\dfrac{\partial U(b)}{\partial A}}{b - \delta\dfrac{\partial V(\underline{\alpha}(b))}{\partial s}},$$

$$\frac{\partial\underline{\beta}(s)}{\partial A} = \frac{\delta\dfrac{\partial V(s)}{\partial A}}{s - \delta\dfrac{\partial U(\underline{\beta}(s))}{\partial b}}.$$

Since

$$\frac{\partial V(\underline{\alpha}(b))}{\partial s} = \int_{\underline{\beta}(\underline{\alpha}(b))}^{\bar{\beta}(\underline{\alpha}(b))}\frac{p_s\theta_s b}{(1-\delta) + p_s\theta_s\delta(G(\bar{\beta}(\underline{\alpha}(b))) - G(\underline{\beta}(\underline{\alpha}(b))))}g(b)db$$

$$\leq \frac{bp_s\theta_s(G(b) - G(\underline{\beta}(\underline{\alpha}(b))))}{(1-\delta) + p_s\theta_s\delta(G(b) - G(\underline{\beta}(\underline{\alpha}(b))))}$$

$$\leq \frac{b}{\delta}\frac{p_s\theta_s\delta(G(b) - G(\underline{\beta}(\underline{\alpha}(b))))}{(1-\delta) + p_s\theta_s\delta(G(b) - G(\underline{\beta}(\underline{\alpha}(b))))}$$

$$< \frac{b}{\delta},$$

and similarly

$$\frac{\partial U(\underline{\beta}(s))}{\partial b} < \frac{s}{\delta}.$$

We conclude that $\partial\underline{\alpha}(b)/\partial A$ has the same sign as $\partial U(b)/\partial A$ and $\partial\underline{\beta}(s)/\partial A$ has the same sign as $\partial V(s)/\partial A$. Q.E.D.

NOTES

1. Roth and Sotomayor (1990) provide an extensive description of existing matching markets.

2. Smith (1993) also considers two-sided heterogeneity in a model without transferable utility. Other related papers include McAfee and McMillan (1988), Morgan (1994), and Sattinger (1993).
3. No termination will be a special case.
4. The utility of a type *b* buyer consuming a type *s* house during all periods can also be specified as *kbs*. All of the results still hold.

REFERENCES

Baye, M. R., and T. F. Cosimano. 1990. "Choosing Sides in Matching Games: Nash Equilibria and Comparative Statics." *Economica* 57 (August):283–293.
Diamond, P. A., and E. Maskin. 1979. "An Equilibrium Analysis of Search and Breach of Contract, I: Steady State." *Bell Journal of Economics* 10:282–316.
McAfee, R. P., and J. McMillan. 1988. "Search Mechanisms." *Journal of Economic Theory* 44:99–123.
McAfee, R. P. 1993. "Mechanism Design by Competing Sellers." *Econometrica* 61:1281–1313.
Morgan, P. B. 1994. "A Model of Search, Coordination and Market Segmentation." Working Paper, State University of New York at Buffalo, May.
Peters, M. 1993. "Equilibrium (Mechanisms) in a Decentralized Market." *Journal of Economic Theory* 64:390–423.
Roth, A. E., and M. A. Oliveira Sotomayor. 1990. "Two-Sided Matching: A Study in Game-Theoretic Modeling and Analysis." *Econometric Society Monograph* No. 18. Cambridge, New York, and Melbourne: Cambridge University Press.
Rubinstein, A. 1982. "Perfect Equilibrium in a Bargaining Model." *Econometrics* 50:97–109.
Rubinstein, A., and A. Wolinsky. 1985. "Equilibrium in Market with Sequential Bargaining." *Econometrica* 53:1133–1150.
Sattinger, M. 1993. "Search and the Efficient Assignment of Workers to Jobs." Working Paper, State University of New York at Albany, August.
Smith, L. A. 1993. "Cross-Sectional Dynamics in a Two-Sided Matching Model." Working Paper, MIT.
Wolinsky, A. 1988. "Dynamic Markets with Competitive Bidding." *Review of Economic Studies* 55:71–84.

LOTTERY QUALIFICATION AUCTIONS

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ABSTRACT

We analyze a Q-lottery qualification auction: the Q highest bidders qualify for a lottery giving each a $1/Q$ chance of obtaining the asset ($Q = 1$ is a second-price auction). We also provide some results for an oral variant, the Q-curtailed oral auction, which sets the price as soon as only Q of n bidders remain in competition, then awards the asset by lottery. Despite the probability of an inefficient outcome, there are many cases in which a seller prefers to choose $Q > 1$. Examples show that Milgrom and Weber's Linkage Principle does not extend to nonstandard auctions. In particular, undermining the privacy of the highest-valuing bidder's information and augmenting expected revenue are seen to be less closely aligned than previous explanations might suggest.

I. INTRODUCTION

In a "lottery qualification" auction, bids are submitted for the purpose of qualifying for a lottery to determine the winner. Specifically, a seller announces a parameter Q, bidders submit sealed bids, and the Q highest bidders qualify for a lottery in which each has a $1/Q$ chance of being the

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